

Relaxed Utility Maximization in Complete Markets

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Abstract

For a relaxed investor – one whose relative risk aversion vanishes as wealth becomes large – the utility maximization problem may not have a solution in the classical sense of an optimal payoff represented by a random variable. This nonexistence puzzle was discovered by Kramkov and Schachermayer (1999), who introduced the *reasonable asymptotic elasticity* condition to exclude such situations.

Utility maximization becomes well-posed again representing payoffs as measures on the sample space, including those allocations singular with respect to the physical probability. The expected utility of such allocations is understood as the maximal utility of its approximations with classical payoffs – the relaxed expected utility.

This paper decomposes relaxed expected utility into its classical and singular parts, represents the singular part in integral form, and proves the existence of optimal solutions for the utility maximization problem, *without* conditions on the asymptotic elasticity. Key to this result is the Polish space structure assumed on the sample space.

Key Words: Utility Maximization, Asymptotic Elasticity, Integral Representation

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Introduction

The problem of maximizing expected utility from a set of payoffs of price x :

$$\max_{p(X)=x} E_P [U(X)] \tag{UM}$$

is central for asset pricing and portfolio choice. If the market is complete (i.e. $p(X) = E_Q[X]$ for some pricing measure Q), the typical solution starts from the Euler equation:

$$U'(X) = y \frac{dQ}{dP} \tag{1}$$

which aligns marginal utility with the state price density dQ/dP . If the Lagrange multiplier y satisfies the saturation condition $E_Q[X] = x$, then the payoff $X^*(y) = (U')^{-1}(y dQ/dP)$ is optimal for the problem (UM).

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This argument is so common that passing from a solution of (1) to a solution of (UM) is considered almost automatic. Checking the condition $E_Q[X] = x$ seems a formality, to be skipped if the actual value of y is not required.

Yet, the argument may fail. For certain combinations of the utility function U , the state price density Q , and the initial capital x , none of the payoffs $X^*(y)$ satisfies $E_Q[X^*(y)] = x$, and the problem (UM) has no solution – a phenomenon first discovered by Kramkov and Schachermayer (1999, Example 5.2). Indeed, they show the existence of a solution under the asymptotic elasticity condition:

$$AE(U) = \limsup_{x \uparrow \infty} \frac{xU'(x)}{U(x)} < 1$$

This condition has a clear interpretation in terms of asymptotic relative risk aversion:

$$ARRA(U) = \lim_{x \uparrow \infty} -\frac{xU''(x)}{U'(x)}$$

When this limit exists, De l'Hôpital's rule implies that the condition $AE(U) < 1$ is equivalent to $ARRA(U) > 0$, i.e. relative risk aversion is bounded away from zero for arbitrarily large wealth levels. Thus, optimal payoffs may not exist for utility functions which are *asymptotically relatively risk neutral*, i.e. $ARRA(U) = 0$.

It is tempting to dismiss such examples as mathematical pathologies without economic substance. Indeed, all common examples of utility functions, including the logarithmic, power, exponential, and in general the HARA (Hyperbolic Absolute Risk Aversion) class, do satisfy $ARRA(U) > 0$. This class of utility functions is ubiquitous in Finance, and a condition violated by this class seems of little interest.

However, power utilities themselves leads to a utility function satisfying $ARRA(U) = 0$ in heterogeneous preferences equilibria. In a model with several agents with individual constant relative risk aversion (i.e. power utility), Benninga and Mayshar (2000) and Cvitanic and Malamud (2008) show that the utility function of the representative agent has decreasing relative risk aversion, which converges – for large levels of wealth – to the value of the least risk averse agent. Thus, the presence of agents with arbitrarily low relative risk aversion implies that $ARRA(U) = 0$.

This paper studies the utility maximization problems for complete markets, relaxing the assumption $AE(U) < 1$. The central idea is that the topological structure on the sample space Ω allows to obtain a solution even in the critical case $AE(U) = 1$. In all models of interest, the sample space Ω is already endowed with such a topology, but the classical theory of utility maximization discards topological information, focusing on the measurable structure alone.

This loss of information is inconsequential if $AE(U) < 1$: then a random variable X that maximizes expected utility always exists. But if $AE(U) = 1$ and the initial x capital exceeds some critical value x^* , then the agent may achieve maximal utility by concentrating capital on events of arbitrarily small probability. Thus, the candidate optimum would allocate finite capital on a set of probability zero. Alas, expected utility neglects null sets, and cannot account for such singular allocation.

The topology on Ω resolves this problem by identifying available payoffs with Radon measures μ of mass less than or equal to x – the space of relaxed payoffs. Then the contribution to expected utility of μ splits into two parts. The classical expected utility $E_P[U(X)]$ accounts for the component $\mu_a = XdQ$, absolutely continuous with respect to Q . The component μ_s , singular with respect to Q leads to the novel term $\int \varphi d\mu_s$, which credits the concentration of capital on null sets for its contribution to expected utility. The “singular utility” φ depends on both the utility function U and on the pricing measure Q .

This paper contributes to Mathematical Finance by resolving the nonexistence puzzle of Kramkov and Schachermayer (1999) in complete markets, proving the existence of a solution in a larger space of payoffs, and it clarifies the structure of the expected utility and its maximizers.

Mathematically, the main result is Theorem 1.3, which can be read as an integral representation result for the utility functional. In comparison with similar results in the literature, applications to Mathematical Finance require more flexible assumptions on the sample space Ω , assumed to be Polish, not necessarily locally compact.

The rest of the paper is organized as follows: Section 1 summarizes the assumptions and the main results, discussing their significance. Section 2 proves the integral representation result, and it is probably the most technical part of the paper. The utility maximization result is proved in Section 3, while the last Section contains examples and counterexamples which show the relevance of the results and their assumptions.

1 Summary of Results

The paper makes the following model assumptions:

Assumption 1.1. *Let (Ω, \mathcal{T}) be a Polish space, and P a Borel-regular probability on the Borel σ -field \mathcal{F} .*

- i) The utility function $U : (0, +\infty) \mapsto (-\infty, +\infty)$ is strictly increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions $U'(0^+) = +\infty$ and $U'(+\infty) = 0$.*
- ii) $\mathcal{C}(x) = \{X \in L^0_+ | E_Q[X] \leq x\}$ for $x > 0$, where Q is equivalent to P .*
- iii) $\sup_{X \in \mathcal{C}(x)} E_P[U(X)] < U(\infty)$*
- iv) P (and hence Q) has full support, i.e. $P(G) > 0$ for any open set G .*

i) means that marginal utility spans the whole range $(0, \infty)$. Appetites change smoothly. *ii)* defines the set of available payoffs in terms on the pricing measure Q , and entails that the market is complete. *iii)* is a well-posedness condition. Bliss utility cannot be reached. *iv)* means that Ω includes only relevant events. It does not restrict generality, in that *iv)* always holds after replacing Ω with the support of P .

The pricing probability Q identifies each classical payoff X with the finite Borel measure $\mu_X = X dQ$, defined by $\mu_X(A) = E_Q[X 1_A]$. With this identification, the expected utility map $X \mapsto I_U(X)$ has the expression:

$$I_U(X) := E_P[U(X)] = \int_{\Omega} U \left(\frac{d\mu_X}{dQ}(\omega) \right) dP(\omega) = \int_{\Omega} U \left(\frac{d\mu_X}{dQ}(\omega) \right) \frac{dP}{dQ}(\omega) dQ(\omega) \quad (2)$$

Kramkov and Schachermayer (1999) show with counterexamples that the original problem (UM) may not have a solution if $AE(U) = 1$. In a complete market, they show that maximizing sequences may entail concentration of capital on “cheap” Arrow-Debreu securities, on which $\frac{dQ}{dP}(\omega) \approx 0$. Such securities, which yield a large payoff X on an event of tiny probability, seem superficially irrelevant for utility maximization, as the marginal utility $U'(X)$ decreases to zero for large payoffs. However, as the odds become more attractive, the quantity $U'(X)dP/dQ$ may remain bounded. For example, imagine a sequence $(\omega_n)_{n \geq 1}$ converging to some ω_{∞} , and such that $\lim_{n \uparrow \infty} (dP/dQ)(\omega_n) \uparrow \infty$. Thus, an optimal allocation, where $U'(X(\omega))$ is proportional to $(dP/dQ)(\omega)$, may well entail a “payoff” such that $Q(\omega_{\infty}) > 0$, even if $P(\omega_{\infty}) = 0$. See Schachermayer (2002) for a further discussion of this phenomenon.

This scenario baffles the existing mathematical theory in two ways. First, the utility map loses its upper semicontinuity with respect to maximizing sequences, as the utilities of maximizing payoffs are no longer uniformly integrable. Second, the purely measure theoretic setting (Ω, \mathcal{F}, P) becomes inadequate to represent singular capital allocations. If a maximizing sequence $(X_n dQ)_{n \geq 0}$ converges to a Dirac delta on some ω , this delta is a natural candidate for a maximizer. On the other hand, if $P(\omega) = 0$, removing ω from the original Ω leads to an equivalent model where no such candidate exists. Thus, a solution may or may not exist, depending on the initial choice of the sample space Ω .

This paper starts from the observation that in most models the sample space Ω is already equipped with a topological structure. For example, in diffusion models Ω is the Wiener space endowed with the uniform topology, while discontinuous models lead to the Skorokhod space. Furthermore, these topologies are compatible with a complete separable metric – they are Polish spaces.

The Polish space structure allows to identify payoffs as measures. This perspective is economically straightforward, thinking of Ω as a roulette table, and of a payoff as a distribution of chips on the various numbers. The payoffs of the form $\mu_X = X dQ$ are a subclass of the norm dual space $(C_b(\Omega))^*$, which is isometric to $rba(\Omega)$, the space of Borel regular, finitely additive signed measures on Ω (Dunford and Schwartz, 1988, IV.6). Each element $\mu \in rba(\Omega)$ admits the unique three-way decomposition:

$$\mu = \mu_a + \mu_s + \mu_p \quad (3)$$

where μ_a and μ_s are countably additive measures, respectively absolutely continuous and singular with respect to P , and μ_p is a purely finitely additive measure. All three components are Borel regular.

Because $rba(\Omega)$ is the dual of a Banach space, its bounded sets – including sequences of available payoffs in $\mathcal{C}(x)$ – are relatively weak star compact. This property is crucial, as it yields limits to maximizing sequences.

Definition 1.2. *A relaxed payoff is an element of $\mathcal{D}(x)$, the weak star $\sigma(rba(\Omega), C_b(\Omega))$ closed set $\{\mu \in rba(\Omega)_+ \mid \mu(\Omega) \leq x\}$.*

The disadvantage of $\mathcal{D}(x)$ is to include purely finitely additive measures, which have a dubious interpretation as payoffs. By contrast, countably additive measures – including those singular with respect to Q – allow the usual Arrow-Debreu interpretation of bets paying off in certain states of nature. This paper resolves this issue by allowing *a priori* all relaxed payoffs, including finitely additive ones. Then, an additional coercivity assumption implies *a posteriori* that the optimal payoff is countably additive. This assumption ensures that exceptionally favorable states (i.e., where φ is high) do not disperse outside the compact sets of Ω .

Defining $\mathcal{D}(x)$ as the set of relaxed payoffs, the relaxed utility map $\overline{I_U} : rba(\Omega) \rightarrow [-\infty, +\infty)$, defined on $rba(\Omega)$, is the upper semicontinuous envelope of the original I_U :

$$\overline{I_U}(\mu) = \inf\{G(\mu) \mid G : rba(\Omega) \rightarrow [-\infty, +\infty), G \text{ weak}^* \text{ u.s.c.}, G \geq I_U \text{ on } L^1(Q)\} \quad (4)$$

Since the relaxed utility map $\overline{I_U}$ is weak star upper semicontinuous by definition, and the space of relaxed payoffs $\mathcal{D}(x)$ is weak star compact, the relaxed utility maximization problem:

$$\max_{\mu \in \mathcal{D}(x)} \overline{I_U}(\mu) \quad (\text{RUM})$$

admits a solution by construction, and the problems (UM) and (RUM) have the same value,

$$\sup_{X \in \mathcal{C}(x)} I_U(X) = \max_{\mu \in \mathcal{D}(x)} \overline{I_U}(\mu)$$

The challenge is to find a “concrete” representation for $\overline{I_U}$, i.e. an explicit formula for the relaxed utility map. This task, which is accomplished in Section 2, involves two additional concepts: the *singular utility* φ , and the *sup-convolution* W . These concepts in turn rely on the convex conjugates of the utility function U and of the expected utility functional I_U , which are discussed in the next Section.

The convex conjugate of U is the function $V : \mathbb{R} \rightarrow (-\infty, +\infty]$ defined as $V(y) = \sup_{x>0}(U(x) - xy)$, so that $V(y) = +\infty$ for $y < 0$. The singular utility is defined as the nonnegative function:

$$\varphi(\omega) = \inf \left\{ g(\omega) \mid g \in \mathcal{C}_b(\Omega), E_P \left[V \left(g \frac{dQ}{dP} \right) < \infty \right] \right\} \quad (5)$$

which is upper semi-continuous, because it is the infimum of a family of continuous functions. Assumption 1.1 iii) implies that φ is finite valued. Indeed, Kramkov and Schachermayer (1999, Theorem 2.0 i)) show that this assumption is equivalent to the existence of some $\tilde{y} > 0$ such that $E_P[V(\tilde{y} \frac{dQ}{dP})] < +\infty$. Thus, $\varphi \leq \tilde{y}$. $W : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is defined as the pointwise *sup-convolution* of the utility function U and of the random function $x \mapsto x\varphi(\omega) \frac{dQ}{dP}(\omega)$:

$$W(\omega, x) := \sup_{z \leq x} \left(U(z) + (x - z)\varphi(\omega) \frac{dQ}{dP}(\omega) \right) \quad (6)$$

The main result on integral representation is then:

Theorem 1.3. *Let $\mu \in rba(\Omega)_+$, and let Q be a probability on Ω with full support.*

i) In general:

$$\overline{I_U}(\mu) = E_P \left[W \left(\cdot, \frac{d\mu_a}{dQ} \right) \right] + \int \varphi d\mu_s + \inf_{f \in \text{Dom}(J_V)} \mu_p(f) \quad (7)$$

ii) If $\varphi = 0$ P -a.s., then

$$\overline{I_U}(\mu) = E_P \left[U \left(\frac{d\mu_a}{dQ} \right) \right] + \int \varphi d\mu_s + \inf_{f \in \text{Dom}(J_V)} \mu_p(f) \quad (8)$$

iii) If $\limsup_{x \uparrow \infty} \frac{xU'(x)}{U(x)} < 1$, then $\{\varphi = 0\} = \Omega$ and

$$\overline{I_U}(\mu) = E_P \left[U \left(\frac{d\mu_a}{dQ} \right) \right] \quad (9)$$

This result is understood as follows. The general formula *i)* holds for any $\mu \in rba(\Omega)_+$, but does not have a sound economic interpretation, since it involves the finitely additive part μ_p and the sup-convolution W , which differs from the original utility function U . Formula *ii)* resolves the second issue, showing that W boils down to U if φ is almost surely null. Example 4.3 in Section 4 shows with a counterexample that U and W may differ without this additional assumption.

Then the relaxed utility is understood as the sum of three parts: the classical expected utility $E[U(X)]$, where $X = \frac{d\mu_a}{dQ}$, the singular utility $\int \varphi d\mu_s$, and the purely finitely additive term μ_p . Intuitively, the singular term accounts for the utility from the concentration of wealth on P -null events, in that $\varphi(\omega)$ represents the maximal expected utility from a Dirac delta concentrated at ω . Indeed, φ vanishes at each ω where dP/dQ is locally bounded (i.e. bounded in a neighborhood of ω), because concentrating wealth is suboptimal if the odds are finite. On the other hand, concentration of wealth may yield a positive utility $\varphi(\omega)$ at those ω where dP/dQ is unbounded, that is, when

the odds are arbitrarily good. The value of $\varphi(\omega)$ depends on the speed at which dP/dQ explodes near ω .

Finally, formula *iii*) reconciles the theorem with the result of Kramkov and Schachermayer (1999), who show that the utility maximization problem admits a classical solution under the asymptotic elasticity assumption $AE(U) = \limsup_{x \uparrow \infty} \frac{xU'(x)}{U(x)} < 1$, which implies that φ is zero everywhere (and not merely almost). Thus, the additional terms vanish, and the expected utility function depends only on $\mu_a = XdQ$.

When $AE(U) = 1$, Assumption 1.1 and $\varphi = 0$ do not guarantee that any optimizer μ^* of (RUM) is a measure, i.e. $\mu_p^* = 0$. Example 4.4 makes this point with a counterexample. This problem is resolved by the next Assumption, which rules out the purely finitely additive part.

Assumption 1.4. *There exist $g \in \text{Dom}(J_V)$ such that the closed set $K = \{g \geq y_0 - \varepsilon\}$ is compact for some $\varepsilon > 0$, where $y_0 = \sup_{\omega \in \Omega} \varphi(\omega)$.*

To state the main result on utility maximization, define u as the value function of the utility maximization problem (UM)

$$u(x) = \sup\{E_P[U(X)] \mid E_Q[X] \leq x\}$$

and let v be its conjugate: $v(y) = \sup_{x>0}\{u(x) - xy\}$. Finally, set $x_0 = \lim_{y \downarrow y_0} -v'(y) = -v'_+(y_0)$. Then, $x_0 \in (0, +\infty]$ is the capital threshold above which the optimal payoff includes a singular component.

Theorem 1.5. *If Assumptions 1.1 and 1.4 hold, and $\varphi = 0$ a.s., it follows that:*

i) $u(x) = \max_{\mu \in \mathcal{D}(x)} \overline{I_U}(\mu)$

ii) $\mu^* = X^*dQ + \mu_s^*$, and:

$$u(x) = E[U(X^*)] + \int \varphi d\mu_s^*$$

X^* is unique, and the budget constraint is binding: $\mu^*(\Omega) = E_Q[X^*] + \mu_s^*(\Omega) = x$. The support of any μ_s^* satisfies:

$$\text{supp}(\mu_s^*) \subseteq \text{argmax}(\varphi)$$

iii) *Optimizers depend on the initial capital x as follows:*

a) $x \leq x_0$ (Kramkov and Schachermayer (1999), Theorem 2.0)

The unique solution is $\mu^* = X^*(x)dQ$, where $X^*(x) = (U')^{-1}\left(y(x)\frac{dQ}{dP}\right)$ and $y(x) = (v')^{-1}(-x)$.

b) $x > x_0$

Any solution has the form $\mu^* = X^*(x)dQ + \mu_s^*$, where $X^*(x) = X^*(x_0) = (U')^{-1}\left(y_0\frac{dQ}{dP}\right)$ and $\mu_s^*(\Omega) = x - x_0$. Therefore $u(x) = u(x_0) + (x - x_0) \max_{\omega} \varphi(\omega) = u(x_0) + (x - x_0)y_0$.

The novelty of this Theorem is the existence of optimal solutions, and their description in the singular case: when x_0 is finite and $x > x_0$, it is optimal to invest the residual capital $x - x_0$ in a very unlikely – but also very favorable – bet μ_s^* . Such bet is not unique in general, because its contribution to expected utility is linear and therefore multiple solutions arise as soon as $\text{argmax}(\varphi)$ has more than one element.

2 Representation of the relaxed functional $\overline{I_U}$

This section proves Theorem 1.3, the representation formula for the relaxed utility map $\overline{I_U}$. The argument proceeds in three steps:

- i) separate in $\overline{I_U}$ the countably additive part from the purely finitely additive part (Lemma 2.2);
- ii) find an integral representation for the countably additive part, separating the absolutely continuous and the singular components with respect to Q (Proposition 2.6);
- iii) identify the absolutely continuous part as the original expected utility map, and the singular part as an “asymptotic utility” (Lemma 2.4 and 2.9).

The convex conjugate $J_V : \mathcal{C}_b(\Omega) \rightarrow (-\infty, +\infty]$ of the expected utility map I_U is:

$$J_V(g) := \sup_{X \in L^1(Q)} (I_U(X) - E_Q[gX]) = E_P \left[V \left(g \frac{dQ}{dP} \right) \right] \quad (10)$$

The domain of J_V is defined as $\text{Dom } J_V = \{g \in \mathcal{C}_b(\Omega) : E[V(gdQ/dP)] < \infty\}$ and is a subset of $\mathcal{C}_b(\Omega)_+$. The next Lemma collects some properties of the conjugate functional J_V .

Lemma 2.1.

- i) $J_V(g) = E_P \left[V \left(g \frac{dQ}{dP} \right) \right]$;
- ii) $\text{Dom}(J_V) = \{g \in \mathcal{C}_b(\Omega) \mid E_P[V(g \frac{dQ}{dP})] < +\infty\}$, the proper domain of J_V , is contained in $\mathcal{C}_b(\Omega)_+$ and is directed downward;
- iii) $\varphi(\omega) = \inf_{g \in \text{Dom}(J_V)} g(\omega)$ defines a random variable, which is positive, bounded, and upper semicontinuous. In addition, there exists a decreasing sequence $(g_k)_{k \geq 1} \subset \text{Dom}(J_V)$ such that $g_k(\omega) \downarrow \varphi(\omega)$ for all ω .

Proof. $L^1(Q)$ is decomposable (i.e. $f1_A + g1_{\Omega \setminus A} \in L^1(Q)$ for any $f, g \in L^1(Q)$ and $A \in \mathcal{F}$), therefore i) follows from (Rockafellar, 1974, Theorem 21, part a)). Since $\text{Dom}(V) \subseteq \mathbb{R}_+$, then $\text{Dom}(J_V) \subset \mathcal{C}_b(\Omega)_+$. Thus, the pointwise infimum φ of the family of continuous, bounded, nonnegative functions $\text{Dom}(J_V)$ is well-defined, nonnegative, bounded and upper semicontinuous. Also, $\text{Dom}(J_V)$ is directed downward, because $g \wedge f \in \text{Dom}(J_V)$ if $g, f \in \text{Dom}(J_V)$:

$$E_P \left[V \left(g \wedge f \frac{dQ}{dP} \right) \right] = E_P \left[V \left(g \frac{dQ}{dP} \right) 1_{\{g \leq f\}} \right] + E_P \left[V \left(f \frac{dQ}{dP} \right) 1_{\{f < g\}} \right] < +\infty$$

Moreover, the space $\mathcal{C}_b(\Omega)$ has the countable supremum property (Aliprantis and Border, 2006, Theorem 8.22). This combined with the directed-downward property implies the existence of a monotone sequence $(g_k)_{k \geq 1}$ in $\text{Dom}(J_V)$ such that $g_k \geq \varphi$ and $g_k \downarrow \varphi$ pointwise. \square

An application of Hahn-Banach separation theorem (see e.g. Borwein and Lewis (2006, Theorem 4.2.8) ensures that the relaxation $\overline{I_U}$ coincides with the biconjugate functional $(I_U)^{**} : rba(\Omega) \rightarrow [-\infty, +\infty)$, which is defined as:

$$(I_U)^{**}(\mu) = \inf_{g \in \mathcal{C}_b(\Omega)} \left(\mu(g) + E_P \left[V \left(g \frac{dQ}{dP} \right) \right] \right) \quad (11)$$

The infimum over $\mathcal{C}_b(\Omega)$ in the above formula can be replaced by the infimum over $\text{Dom}(J_V)$, since by (11), $(I_U)^{**} = -\infty$ whenever μ is not positive. The results in the rest of the Section are stated only for $\mu \in rba_+$.

The following Lemma proves the first part of Theorem 1.3, which states that the relaxation is additive across the Yosida and Hewitt (1952) decomposition of $\mu = \mu_c + \mu_p$ in terms of the countably additive part $\mu_c = \mu_a + \mu_s$, and the purely finitely additive part μ_p . Since Ω is a Polish space, any measure $\mu = \mu_c \in rba(\Omega)_+$ is a Radon measure, that is compact-inner regular (Aliprantis and Border, 2006, Theorem 12.7). By contrast, any purely finitely additive $\mu = \mu_p$ vanishes on compact sets (Aliprantis and Border, 2006, Theorem 12.4). This contrasting behavior allows the separation of the contributes of μ_c and μ_p in the relaxation (12).

Lemma 2.2. *Let $\mu \in rba(\Omega)_+$. Then*

$$\overline{I_U}(\mu) = \overline{I_U}(\mu_c) + \inf_{f \in \text{Dom}(J_V)} \mu_p(f) \quad (12)$$

Proof. The inequality \geq follows from $\overline{I_U} = (I_U)^{**}$ and from the inequality

$$E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu(g) \geq E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu_c(g) + \inf_{f \in \text{Dom}(J_V)} \mu_p(f) \geq (I_U)^{**}(\mu_c) + \inf_{f \in \text{Dom}(J_V)} \mu_p(f)$$

For the opposite inequality, note that $(P + \mu_c)$ is a Radon measure. Hence there exists an increasing sequence of compact sets K^n such that $(P + \mu_c)(\Omega \setminus K^n) < \frac{1}{n}$. By contrast, $\mu_p(K^n) = 0$ for all n because μ_p is purely finitely additive. Thus μ_p is concentrated on $\Omega \setminus K^n$. Borel-regularity of μ_p implies the existence of closed sets $C^n \subseteq \Omega \setminus K^n$ such that

$$\mu_p(\Omega \setminus C^n) < \frac{1}{n}$$

In the Polish space Ω , closed sets are separated by continuous functions. That is, there exists a continuous function $\alpha^n : \Omega \rightarrow [0, 1]$ which is equal to 1 on K^n and 0 on C^n . In fact, if d is a distance that induces the topology \mathcal{T} on Ω , one such function is:

$$\alpha^n(\omega) = \frac{d(\omega, C^n)}{d(\omega, C^n) + d(\omega, K^n)}$$

Up to a subsequence, α^n converges to 1 $(P + \mu_c)$ -a.s. Fix an some $f, g \in \text{Dom}(J_V)$, and set

$$h^n = \alpha^n g + (1 - \alpha^n) f$$

Convexity of V and boundedness of α^n imply $h^n \in \text{Dom}(J_V)$, because

$$E_P \left[V \left(h^n \frac{dQ}{dP} \right) \right] + \mu(h^n) \leq E_P \left[\alpha^n V \left(g \frac{dQ}{dP} \right) \right] + E_P \left[(1 - \alpha^n) V \left(f \frac{dQ}{dP} \right) \right] + \mu(h^n) < +\infty$$

Also, since $h^n - f = \alpha^n(g - f)$, and $0 \leq \alpha^n \leq 1$

$$\mu_p(h^n) \leq \frac{1}{n} \|g - f\|_\infty + \mu_p(f)$$

It follows that

$$\begin{aligned} (I_U)^{**}(\mu) &\leq E_P \left[V \left(h^n \frac{dQ}{dP} \right) \right] + \mu(h^n) \\ &\leq E_P \left[\alpha^n V \left(g \frac{dQ}{dP} \right) \right] + E_P \left[(1 - \alpha^n) V \left(f \frac{dQ}{dP} \right) \right] + \mu_c(h^n) + \frac{1}{n} \|g - f\|_\infty + \mu_p(f) \end{aligned}$$

and passing to the liminf,

$$\begin{aligned}
(I_U)^{**}(\mu) &\leq \liminf_{n \uparrow \infty} E_P \left[V \left(h^n \frac{dQ}{dP} \right) \right] + \mu(h^n) \\
&\leq \lim_{n \uparrow \infty} \left\{ E_P \left[\alpha^n V \left(g \frac{dQ}{dP} \right) \right] + E_P \left[(1 - \alpha^n) V \left(f \frac{dQ}{dP} \right) \right] + \mu_c(h^n) + \frac{1}{n} \|g - f\|_\infty + \mu_p(f) \right\} \\
&= E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu_c(g) + \mu_p(f)
\end{aligned}$$

where the liminf in the second line becomes a limit, because α^n is bounded and converges to 1 ($P + \mu_c$)-a.s., hence the dominated convergence theorem applies. Thus:

$$(I_U)^{**}(\mu) \leq \inf_{f, g \in \text{Dom}(J_V)} \left(E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu_c(g) + \mu_p(f) \right) = (I_U)^{**}(\mu_c) + \inf_{f \in \text{Dom}(J_V)} \mu_p(f)$$

which completes the proof. \square

Remark 2.3. It is tempting to replace the expression $\inf_{f \in \text{Dom}(J_V)} \mu_p(f)$ with the simpler $\mu_p(\varphi)$, i.e exchange the infimum and the expectation. However, since μ_p is not countably additive, only the inequality $\mu_p(\varphi) \leq \inf_{f \in \text{Dom}(J_V)} \mu_p(f)$ holds in general. Example 4.4 shows a situation where $\varphi = 0$ but $\inf_{f \in \text{Dom}(J_V)} \mu_p(f) > 0$.

Denote the countably additive elements of $rb\alpha(\Omega)_+$ simply by \mathcal{M}_+ , the subset of positive Radon measures. The next step is to prove an integral representation for $\overline{I_U}(\mu)$ when $\mu \in \mathcal{M}_+$. This result extends in part the work of Bouchitté and Valadier (1988), who consider a locally compact space Ω . Relaxing this assumption is central in Mathematical Finance where sample spaces are typically infinite-dimensional.

Recall the definition of $W : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$, the ω -wise *sup-convolution* of the utility function U and of the random function $x \mapsto x\varphi(\omega) \frac{dQ}{dP}(\omega)$:

$$W(\omega, x) := \sup_{z \leq x} \left(U(z) + (x - z)\varphi(\omega) \frac{dQ}{dP}(\omega) \right) \quad (13)$$

The sup-convolution W may differ from U only on the event $\left\{ \varphi \frac{dQ}{dP} > 0 \right\}$:

Lemma 2.4. $\{\omega \mid W(\omega, x) = U(x) \text{ for all } x > 0\} = \left\{ \omega \mid \varphi(\omega) \frac{dQ}{dP}(\omega) = 0 \right\}$.

Proof. If $\varphi(\omega) \frac{dQ}{dP}(\omega) = 0$, then $W(\omega, x) = U(x)$ from the definition of W . Viceversa, observe that if $W(\omega, x) = U(x)$ for all $x > 0$, then:

$$U'(z) - \varphi(\omega) \frac{dQ}{dP}(\omega) \geq 0 \quad \text{for all } z > 0$$

and the claim follows from the Inada condition $U'(\infty) = 0$:

$$0 = \lim_{z \rightarrow +\infty} U'(z) \geq \varphi(\omega) \frac{dQ}{dP}(\omega)$$

\square

Lemma 2.5. *If $\mu \in \mathcal{M}_+$, then*

$$\overline{I_U}(\mu) = \sup_{X_n \xrightarrow{*} \mu} \limsup_{n \uparrow \infty} I_U(X_n)$$

where the maximum is taken over all sequences $(X_n)_n$ that weak star converge to μ .

Proof. The relaxation $\overline{I_U}$ is defined as the upper semicontinuous envelope of I_U , hence (cf. Buttazzo (1989, Proposition 1.3.1)):

$$\overline{I_U}(\mu) = \sup_{X_\alpha \xrightarrow{*} \mu} \limsup_{\alpha} I_U(X_\alpha)$$

where the supremum is taken over all nets $(X_\alpha)_{\alpha \in I}$ converging weak star to μ . Since the trace of the weak star topology on norm bounded subset of \mathcal{M}_+ is metrizable (e.g. by the Dudley distance, cf. Ambrosio, Gigli and Savaré (2008, Section 5.1)), nets can be replaced by sequences for $\mu \in \mathcal{M}_+$. \square

Proposition 2.6. *Let $\mu \in \mathcal{M}_+$, so that $\mu = \mu_c = \mu_a + \mu_s$. Then*

$$\overline{I_U}(\mu) = E_P \left[W \left(\cdot, \frac{d\mu_a}{dQ} \right) \right] + \int \varphi d\mu_s \quad (14)$$

Proof. By (11) and Lemma 2.5, the relaxed functional satisfies:

$$\overline{I_U}(\mu) = \sup_{X_n \xrightarrow{*} \mu} \limsup_{n \uparrow \infty} I_U(X_n) = (I_U)^{**}(\mu) = \inf_{g \in \mathcal{C}_b(\Omega)} \left(\mu(g) + E_P \left[V \left(g \frac{dQ}{dP} \right) \right] \right)$$

Consider a maximizing sequence $(X_n)_{n \geq 1}$ for $\overline{I_U}(\mu)$. As $\mu^n := X_n dQ$ converges to μ in the weak star topology, $(X_n)_n$ is bounded in $L^1(Q)$. Up to a sequence of convex combinations, which preserves the maximizing property by concavity of I_U , Komlos Theorem implies that $(X_n)_{n \geq 1}$ converges Q -a.s. to some positive random variable Z . Lemma 2.7 below implies that $Z \leq \frac{d\mu_a}{dQ}$. For any $g \in \text{Dom}(J_V)$, the pointwise Fenchel inequality $U(x) \leq xy + V(y)$ yields:

$$U(X_n) - X_n g \frac{dQ}{dP} \leq V \left(g \frac{dQ}{dP} \right)$$

Passing to the limsup of the expectations, Fatou's Lemma implies that:

$$\limsup_{n \uparrow \infty} E_P \left[U(X_n) - X_n g \frac{dQ}{dP} \right] \leq E_P \left[U(Z) - Z g \frac{dQ}{dP} \right] \leq E_P \left[V \left(g \frac{dQ}{dP} \right) \right]$$

Since $(X_n)_{n \geq 1}$ is maximizing, and $E[X_n g \frac{dQ}{dP}] = E_Q[X_n g]$ converges to $\mu(g)$,

$$\overline{I_U}(\mu) - \mu(g) \leq E_P \left[U(Z) - Z g \frac{dQ}{dP} \right] \leq E_P \left[V \left(g \frac{dQ}{dP} \right) \right]$$

or equivalently, adding $\mu(g)$ to all members above and decomposing $\mu = \mu_a + \mu_s$

$$\overline{I_U}(\mu) \leq E_P \left[U(Z) + \left(\frac{d\mu_a}{dQ} - Z \right) g \frac{dQ}{dP} \right] + \mu_s(g) \leq E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu(g)$$

which holds for any $g \in \text{Dom}(J_V)$. Take now the infimum on g in the above chain. Lemma 2.1 iii) and Monotone Convergence Theorem ensure that the infimum can be taken within the expectation signs in the middle term. Then

$$\overline{I_U}(\mu) \leq E_P \left[U(Z) + \left(\frac{d\mu_a}{dQ} - Z \right) \varphi \frac{dQ}{dP} \right] + \mu_s(\varphi) \leq (I_U)^{**}(\mu) \quad (15)$$

which implies that both inequalities are in fact equalities. Thus, it remains to prove that:

$$E_P \left[U(Z) + \left(\frac{d\mu_a}{dQ} - Z \right) \varphi \frac{dQ}{dP} \right] = E_P \left[W \left(\cdot, \frac{d\mu_a}{dQ} \right) \right]$$

For any $g \in \text{Dom}(J_V)$ and $0 \leq z \leq x$:

$$U(z) + (x-z)\varphi \frac{dQ}{dP} \leq U(z) + (x-z)g \frac{dQ}{dP} \leq V \left(g \frac{dQ}{dP} \right) + xg \frac{dQ}{dP}$$

where the first inequality is due to $\varphi \leq g$ and the second is an application of Fenchel inequality, $U(z) - zy \leq V(y)$. Therefore:

$$U(z) + (x-z)\varphi \frac{dQ}{dP} \leq W(\omega, x) = \sup_{z \leq x} \left(U(z) + (x-z)\varphi \frac{dQ}{dP} \right) \leq V \left(g \frac{dQ}{dP} \right) + xg \frac{dQ}{dP}$$

Substituting x with $\frac{d\mu_a}{dQ}$, z with Z in the first term on the left, and taking expectations:

$$E_P \left[U(Z) + \left(\frac{d\mu_a}{dQ} - Z \right) \varphi \frac{dQ}{dP} \right] \leq E_P \left[W \left(\cdot, \frac{d\mu_a}{dQ} \right) \right] \leq E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu_a(g) \quad (16)$$

Thus, combining (16) with (15), the following holds for any $g \in \text{Dom}(J_V)$:

$$\begin{aligned} \overline{I_U}(\mu) &= E_P \left[U(Z) + \left(\frac{d\mu_a}{dQ} - Z \right) \varphi \frac{dQ}{dP} \right] + \mu_s(\varphi) \\ &\leq E_P \left[W \left(\cdot, \frac{d\mu_a}{dQ} \right) \right] + \mu_s(\varphi) \leq E_P \left[V \left(g \frac{dQ}{dP} \right) \right] + \mu_a(g) + \mu_s(g) \end{aligned}$$

whence the conclusion (14). Moreover $U(Z) + \left(\frac{d\mu_a}{dQ} - Z \right) \varphi \frac{dQ}{dP} = W \left(\cdot, \frac{d\mu_a}{dQ} \right)$ almost surely, whence the pointwise limit Z of the maximizing $(X_n)_n$ verifies

$$Z = \frac{d\mu_a}{dQ} \wedge (U')^{-1} \left(\varphi \frac{dQ}{dP} \right) \quad a.s.$$

□

Lemma 2.7. *Let $(X_n)_{n \geq 1}$ be a bounded sequence in $L^1_+(Q)$, such that X_n converges to X almost surely, and weak star to $\mu \in rba(\Omega)$. Then $X \leq \frac{d\mu_a}{dQ}$ almost surely.*

Proof. Note first that $\mu \geq 0$, $X \geq 0$ and $X \in L^1(Q)$ by Fatou's Lemma. By the compact-inner regularity of the measure $\mu_a + \mu_s$, it suffices to show that:

$$E_Q[I_K X] \leq (\mu_a + \mu_s)(K) \text{ for all compact sets } K$$

Indeed, since the inequality for all compact sets K , it also holds for all Borel sets B , whence $E_Q[I_B X] \leq (\mu_a + \mu_s)(B)$, and in particular $X \leq \frac{d\mu_a}{dQ}$ Q -a.s.

To this end, proceed similarly to the first part of the proof of Proposition 2.2. Consider a compact K . For any $h \geq 1$ there exists a closed set $C_h \subseteq K^c$ with $\mu_p(C_h) \geq \mu_p(\Omega) - \frac{1}{h}$. Also, there is a continuous function g_h^K such that $0 \leq g_h^K \leq 1$, $g_h^K = 1$ on K , $g_h^K = 0$ on C_h and hence, as $h \uparrow +\infty$, $g_h^K \rightarrow 1_K$ pointwise. Then, for all $h \geq 1$:

$$E_Q[I_K X] \leq E_Q[g_h^K X] \leq \lim_{n \uparrow \infty} E_Q[g_h^K X_n] = \mu(g_h^K)$$

where the second inequality is a consequence of Fatou's Lemma, while the equality follows from weak star convergence of $X_n dQ$ to μ . By construction, $\mu_p(g_h^K) \leq \mu_p(\Omega \setminus C_h) \leq \frac{1}{h}$ whence

$$E_Q[I_K X] \leq (\mu_a + \mu_s)(g_h^K) + \frac{1}{h}$$

and the conclusion follows passing to the limit as $h \uparrow \infty$. \square

Remark 2.8. The inequality $X \leq \frac{d\mu_a}{dQ}$ can be strict. Ball and Murat (1989, Example 2) give an example in which $X = 0$ and $\frac{d\mu_a}{dQ} = 1$.

It only remains now to put the pieces together.

Proof of Theorem 1.3. *i)* follows from Lemma 2.2 and Proposition 2.6. $\varphi = 0$ a.s. implies that $W(\omega, x) = U(x)$ almost surely, whence *ii)* follows from *i)* and Lemma 2.4.

To show *iii)*, recall that if $AE(U)$ holds, $E_P[V(ydQ/dP)] < +\infty$ for all constants $y > 0$ (Kramkov and Schachermayer, 2003, Note 2). Then $\varphi = 0$ *everywhere* on Ω , hence both $\int \varphi d\mu_s$ and $\inf_{f \in \text{Dom}(J_V)} \mu_p(f)$ vanish:

$$\inf_{f \in \text{Dom}(J_V)} \mu_p(f) \leq \inf_{y > 0} \mu_p(y) = \mu_p(\Omega) \inf_{y > 0} y = 0$$

\square

Denote by $F \subset \Omega$ the set where $\frac{dP}{dQ}$ is essentially locally bounded (i.e. in a neighborhood):

$$F := \left\{ \omega \left| \text{ess sup}_{\omega' \in U} \frac{dP}{dQ}(\omega') < \infty \text{ for some open } U \ni \omega \right. \right\}$$

The complementary set $F^c := \Omega \setminus F$ is the set of the poles of $\frac{dP}{dQ}$, the points at which $\frac{dP}{dQ}$ is unbounded. By definition, F is open, so F^c is closed. The following proposition shows that φ may be positive only on poles.

Lemma 2.9. $F \subset \{\varphi = 0\}$ hence $\{\varphi > 0\} \subset F^c$.

Proof. If $\omega^* \in F$, there exists an open ball $B(\omega^*, \varepsilon) \subset F$ such that $\frac{dP}{dQ} \leq m$ a.s. on $B(\omega^*, \varepsilon)$. Consider $\tilde{y} > 0$ large enough, so that $E_P \left[V \left(\tilde{y} \frac{dQ}{dP} \right) \right] < \infty$, and for any $y \in (0, \tilde{y})$ consider the continuous bounded function $g_y = y\alpha + \tilde{y}(1 - \alpha)$, where

$$\alpha(\omega) = \frac{d(\omega, \Omega \setminus B(\omega^*, \varepsilon))}{d(\omega, \overline{B}(\omega^*, \frac{\varepsilon}{2})) + d(\omega, \Omega \setminus B(\omega^*, \varepsilon))} \quad (17)$$

Since $\alpha \in [0, 1]$ by construction $y \leq g_y < \tilde{y}$. In addition, $g_y(\omega) = y$ for $\omega \in \overline{B}(\omega^*, \varepsilon/2)$ and $g_y(\omega) = \tilde{y}$ for $\omega \in \Omega \setminus B(\omega^*, \varepsilon)$. To prove that $g_y \in \text{Dom}(J_V)$, split the integral $J_V(g) = E \left[V \left(g_y \frac{dQ}{dP} \right) \right]$ as:

$$E \left[V \left(g_y \frac{dQ}{dP} \right) I_{B(\omega^*, \varepsilon)} \right] + E \left[V \left(g_y \frac{dQ}{dP} \right) I_{\Omega \setminus B(\omega^*, \varepsilon)} \right] \leq V \left(\frac{y}{m} \right) P(B(\omega^*, \varepsilon)) + V(\tilde{y}) P(\Omega \setminus B(\omega^*, \varepsilon))$$

where the inequality holds since V is decreasing and $g_y \frac{dQ}{dP} \geq \frac{y}{m}$ on $B(\omega^*, \varepsilon)$. By definition of φ :

$$\varphi(\omega^*) = \inf_{g \in \text{Dom}(J_V)} g(\omega^*)$$

and from $g_y(\omega^*) = y$, the conclusion $\varphi(\omega^*) = 0$ follows. \square

Corollary 2.10. *If $P(F^c) = 0$, then $\varphi = 0$ a.s.*

3 Proof of Theorem 1.5

Recall that the function $\mathbb{R}_+ \ni y \mapsto v(y) = E \left[V \left(y \frac{dQ}{dP} \right) \right]$ is the restriction of J_V to the constant functions and recall that $x_0 = \lim_{y \downarrow y_0} -v'(y) = -v'_+(y_0)$. The next Lemma shows an alternative characterization of $u(x)$:

Lemma 3.1.

$$\inf_{g \in \mathcal{C}_b(\Omega)} (J_V(g) + \|g\|_\infty x) = \inf_{y > 0} (xy + v(y)) = u(x)$$

Proof. Only the left equality needs a proof, the other one following from Kramkov and Schachermayer (1999). The inequality \leq is obvious. To see the reverse inequality, observe that $\text{Dom}(J_V) \subseteq \mathcal{C}_b(\Omega)_+$ and V is decreasing. Thus, for all $g \geq 0$

$$J_V(g) + x\|g\|_\infty \geq v(\|g\|_\infty) + x\|g\|_\infty \quad (18)$$

which completes the proof. \square

Lemma 3.2. *Let $\mathcal{D}(x) = \{\mu \in rba(\Omega)_+ \mid \mu(\Omega) \leq x\}$, and let $\delta_{\mathcal{D}(x)}$ be the indicator of $\mathcal{D}(x)$. Then its conjugate and biconjugate satisfy:*

$$\begin{aligned} (\delta_{\mathcal{D}(x)})^*(g) &= \sup_{\mu \in rba(\Omega)} \{\mu(g) - \delta_{\mathcal{D}(x)}\} = \|g^+\|_\infty x \\ (\delta_{\mathcal{D}(x)})^{**}(\mu) &= \delta_{\mathcal{D}(x)}(\mu) \end{aligned}$$

Proof. As $\mu(g) \leq \mu(g^+)$ for any positive μ , the supremum in the formula for the conjugate is reached on the μ in $\mathcal{D}(x)$ with support contained in $\{g \geq 0\}$. Thus, without loss of generality suppose $g \geq 0$. The inequality $(\delta_{\mathcal{D}(x)})^*(g) \leq \|g\|_\infty x$ follows from the definition of $\mathcal{D}(x)$. To show that equality holds, fix an arbitrary $\epsilon > 0$. The upper level set $A = \{g > \|g\|_\infty - \epsilon\}$ is open. Since Q has full support, $Q(A) > 0$. Then $\tilde{\mu} = x \frac{1_A}{Q(A)} dQ \in \mathcal{C}(x) \subseteq \mathcal{D}(x)$ and $\tilde{\mu}(g) > (\|g\|_\infty - \epsilon)$, whence $(\delta_{\mathcal{D}(x)})^*(g) \geq (\|g\|_\infty - \epsilon)$ for all ϵ .

The original convex functional $\delta_{\mathcal{D}(x)}$ is already weak star lower semicontinuous, because $\mathcal{D}(x)$ is weak star closed. Therefore it coincides with its lower semicontinuous envelope $(\delta_{\mathcal{D}(x)})^{**}$. \square

Assumption 1.4 is used for the first time in the Lemma below.

Lemma 3.3. *If Assumption 1.4 holds, then $\text{argmax } \varphi$ is compact, and:*

$$y_0 = \max_{\omega \in \Omega} \varphi(\omega) \quad (19)$$

Proof. Set $c = \inf_k \|g_k\|_\infty$, where $(g_k)_{k \geq 1}$ decreases to φ , which exists by Lemma 2.1 *iii*). As shown in (18), $\|g_k\|_\infty \in \text{Dom}(J_V)$ for all k , so

$$c \geq y_0 \geq \sup_{\omega \in \Omega} \varphi(\omega)$$

where the last inequality follows from the definitions of y_0 and φ . To prove (19), we show that $c = \max \varphi$. Up to replacing $(g_k)_{k \geq 1}$ with $(g \wedge g_k)_{k \geq 1}$, assume that the g in Assumption 1.4 is one of the g_k , say g_{k^*} . Then there is a compact upper level of g_{k^*} of the form $K = \{g_{k^*} \geq y_0 - \epsilon^*\}$. As $c \geq y_0$, K contains the closed set $K^* = \{g_{k^*} \geq c - \epsilon^*\}$, which is in turn compact. Outside K^*

$$\varphi \leq g_{k^*} < c - \epsilon^* \quad (20)$$

As K^* is compact and φ is u.s.c., it attains its maximum on K^* . K^* contains all the non empty, closed sets with the finite intersection property: $V_{k,\epsilon} = \{g_k \geq c - \epsilon\}$ for all $k \geq k^*$, $\epsilon < \varepsilon^*$. Therefore their intersection $Y := \bigcap_{k,\epsilon} V_{k,\epsilon}$ is not empty, compact and consists of all the points ω^* where $\lim_k g_k(\omega^*) = c$. But $\lim_k g_k(\omega^*) = \varphi(\omega^*)$, so $c = y_0 = \max_{\omega \in \Omega} \varphi$ and $Y = \operatorname{argmax} \varphi$. \square

Proof of Theorem 1.5. i): It suffices to show that:

$$\inf_{g \in \mathcal{C}_b(\Omega)} (J_V(g) + \|g\|_\infty x) = \max_{\mu \in \mathcal{D}(x)} \overline{I_U}(\mu)$$

Then the claim follows from Lemma 3.1 and the duality formula $u(x) = \inf_{y>0} \{xy + v(y)\}$ (cf. (Kramkov and Schachermayer, 1999, Theorem 2.0)). Since $\operatorname{Dom}(J_V) \subseteq \mathcal{C}_b(\Omega)_+$,

$$\inf_{g \in \mathcal{C}_b(\Omega)} \{J_V(g) + \|g\|_\infty x\} = \inf_{g \in \mathcal{C}_b(\Omega)} \{J_V(g) + \|g^+\|_\infty x\} = \inf_{g \in \mathcal{C}_b(\Omega)} \{J_V(g) + (\delta_{\mathcal{D}(x)})^*(g)\}$$

where the last equality follows by Lemma 3.2.

This Lemma and the Fenchel Theorem (Brezis, 1983, Chapter 1) yield the identity:

$$\inf_{g \in \mathcal{C}_b(\Omega)} (J_V(g) + (\delta_{\mathcal{D}(x)})^*(g)) = \max_{\mu \in \mathcal{D}(x)} \overline{I_U}(\mu)$$

In fact, the Fenchel Theorem implies that

$$\inf_{g \in \mathcal{C}_b(\Omega)} (J_V(g) + (\delta_{\mathcal{D}(x)})^*(g)) = \max_{\mu \in rba(\Omega)} (-(J_V)^*(-\mu) - (\delta_{\mathcal{D}(x)})^{**}(\mu))$$

Now, by definition $(J_V)^*(\mu) = \sup_{g \in \mathcal{C}_b(\Omega)} \{\mu(g) - J_V(g)\}$ and thus $-(J_V)^*(-\mu) = (I_U)^{**}(\mu) = \overline{I_U}(\mu)$, while $(\delta_{\mathcal{D}(x)})^{**}(\mu) = \delta_{\mathcal{D}(x)}(\mu)$.

ii): The constraint is binding because $\overline{I_U}$ is monotone. To prove that any optimal μ^* must be a measure, consider the formula

$$\overline{I_U}(\mu^*) = E_P \left[U \left(\frac{d\mu_a^*}{dQ} \right) \right] + \int \varphi d\mu_s^* + \inf_{f \in \operatorname{Dom}(J_V)} \mu_p^*(f)$$

Suppose that $\mu_p^* \neq 0$, say $0 < \mu_p^*(\Omega) = x' \leq x$. Using (20), the contribution of the purely finitely additive μ_p^* to the (optimal) value $\overline{I_U}(\mu^*)$ is bounded above by:

$$\inf_{f \in \operatorname{Dom}(J_V)} \mu_p^*(f) \leq \mu_p^*(g_{k^*}) = \mu_p^*(g_{k^*} I_{\Omega \setminus K^*}) \leq (y_0 - \varepsilon^*)x'$$

Thus a redistribution of capital, e.g. the measure $\tilde{\mu} = \mu_a^* + \mu_s^* + x'\nu_s$, where ν_s is any probability with support contained in the set $\operatorname{argmax} \varphi$, gives a higher utility:

$$\begin{aligned} \overline{I_U}(\tilde{\mu}) &= E_P \left[U \left(\frac{d\mu_a^*}{dQ} \right) \right] + \int \varphi d(\mu_s^* + x'\nu_s) \\ &= E_P \left[U \left(\frac{d\mu_a^*}{dQ} \right) \right] + \int \varphi d\mu_s^* + y_0 x' \geq \overline{I_U}(\mu^*) + \varepsilon^* x' > \overline{I_U}(\mu^*) \end{aligned}$$

which is a contradiction. Also, $X^*(x) = \frac{d\mu_a^*}{dQ}$ is unique since U is strictly convex. Finally, a monotonicity argument shows that the support of any optimal μ_s^* is contained in $\operatorname{argmax} \varphi$.

iii): The dual problem $\inf_{y>0} (v(y) + xy)$ admits a unique minimizer $y(x)$ for all fixed $x > 0$.

a) $x \leq x_0$.

$y(x)$ is the unique solution of the equation $-v'(y) = x$, i.e. $-E_P \left[\frac{dQ}{dP} V'(y(x)) \frac{dQ}{dP} \right] = x$. Setting $X^*(x) = -V'(y(x) \frac{dQ}{dP}) = (U')^{-1}(y(x) \frac{dQ}{dP})$, Fenchel equality yields

$$U(X^*(x)) = V \left(y(x) \frac{dQ}{dP} \right) + y(x) \frac{dQ}{dP} X^*(x)$$

whence $E[U(X^*(x))] = v(y(x)) + xy(x)$. From $u(x) = \inf_{y>0} \{v(y) + xy\}$, $X^*(x) \in \mathcal{C}(x)$ is the unique optimal payoff.

b) $x > x_0$.

The minimizer of the dual problem is constant, $y(x) = y(x_0) = y_0$. Setting $X^*(x) = X^*(x_0) = -V' \left(y_0 \frac{dQ}{dP} \right) = (U')^{-1}(y_0 \frac{dQ}{dP})$, now $E_Q[X^*(x)] = x_0 \leq x$. An application of Fenchel equality again yields

$$U(X^*(x)) = U(X^*(x_0)) = V(y_0 \frac{dQ}{dP}) + y_0 \frac{dQ}{dP} X^*(x_0)$$

Taking expectations, $E_P[U(X^*(x))] = v(y_0) + x_0 y_0$. Then

$$u(x) = \inf_{y>0} (v(y) + xy) = v(y_0) + x y_0 = E[U(X^*(x))] + y_0(x - x_0)$$

By *ii*) above, any optimal μ_s^* must satisfy $\mu_s^*(\Omega) = x - x_0$.

□

Corollary 3.4. *If $v(y) < +\infty$ for all $y > 0$ (in particular if $AE(U) < 1$), then $y_0 = 0$ and $x_0 = v'_+(0) = +\infty$. So the optimal solution is of the form $\mu = XdQ$ for all $x > 0$.*

Proof. The Inada condition $U'(0) = +\infty$ imply that $V'(0) = -\infty$, whence:

$$x_0 = \lim_{y \downarrow 0} -v'(y) = \lim_{y \downarrow 0} -E_P \left[\frac{dQ}{dP} V' \left(y \frac{dQ}{dP} \right) \right] = +\infty$$

The thesis follows from Theorem 1.5 *iii*). □

Remark 3.5. The Corollary shows that $y_0 = 0$ implies that $x_0 = +\infty$. However, the reverse implication fails, see Example 4.2.

4 Examples and Counterexamples

The examples below explain the role of the singular utility function φ , and the role of the condition $\varphi = 0$ a.s. and Assumption 1.4. The utility function U used is the one defined implicitly by its conjugate $V(y) = e^{1/y}$ for $y > 0$ and $-\infty$ otherwise. Thus, $U(x) = \inf_{y>0} (V(y) + xy) = V(\hat{y}) + x\hat{y}$, where \hat{y} is the unique solution to the equation $V'(y) = -x$, i.e. $e^{1/y}/y^2 = x$. Since this is a transcendental equation, U does not admit a simple expression in terms of elementary functions.

Nevertheless, U satisfies the Inada conditions because $V'(0) = -\infty$ and $V'(\infty) = 0$. Similarly, $U(0) = 1$ and $U(\infty) = \infty$ because $V(0) = \infty$ and $V(\infty) = 1$. Finally, U has asymptotic elasticity equal to 1:

$$\lim_{x \uparrow \infty} \frac{xU'(x)}{U(x)} = \lim_{y \downarrow 0} - \frac{V'(y)y}{V(y) - yV'(y)} = 1 \quad (21)$$

and therefore it violates the assumptions of Kramkov and Schachermayer (1999). Since it is also twice-differentiable, de l'Hôpital's rule implies that U is asymptotically risk-neutral, i.e.:

$$\lim_{x \uparrow \infty} -\frac{xU''(x)}{U'(x)} = 0 \quad (22)$$

Example 4.1 (Infinitely many primal solutions). Consider a bounded double sequence $(\omega_k)_{k \in \mathbb{Z}_0} \subseteq \mathbb{R}$, with downward limit $\omega_{-\infty}$ and upward limit $\omega_{+\infty}$, and set $\Omega = (\omega_k)_{k \in \mathbb{Z}_0} \cup \{\omega_{-\infty}, \omega_{+\infty}\}$. Endowed with the Euclidean topology, Ω is a compact Polish space. Define P by $P(\omega_k) = c_1|k|^{-3}e^{-|k|}$ and $P(\omega_{-\infty}) = P(\omega_{+\infty}) = 0$, and set $\frac{dQ}{dP}(\omega_k) = c_2/|k|$ where c_1, c_2 are normalizing constants.

A simple calculation shows that $v(y) = E[V(y\frac{dQ}{dP})]$ is finite iff $y \geq 1/c_2 = y_0$, and that $v'_+(y_0)$ is also finite and equal to $-2c_1c_2 \sum_{n \geq 1} 1/n^2$. In particular, the no-bliss condition $\sup_{X \in \mathcal{C}(x)} E_P[U(X)] < \infty$ is satisfied. Moreover, since $\frac{dP}{dQ}$ is finite on \mathbb{Z}_0 , and any $\omega_k \in \mathbb{Z}_0$ is an isolated point, it follows that $\varphi(\omega_k) = 0$ for all $k \in \mathbb{Z}_0$, while $\varphi(\omega_{-\infty}) = \varphi(\omega_{+\infty}) = y_0$. Thus, $\{\varphi > 0\} = \{\omega_{-\infty}, \omega_{+\infty}\}$ is a negligible set and $\varphi = 0$ a.s. holds. Assumption 1.4 holds since Ω is compact.

For $x \leq x_0 = -v'_+(y_0)$, the problem admits a classical solution $X^*(\omega_k) = \frac{d\mu_a^*}{dQ}(\omega_k)$, identified by the system:

$$\begin{aligned} U'(X^*(\omega_k)) &= y \frac{dQ}{dP}(\omega_k) & k \in \mathbb{Z}_0 \\ \sum_{k \in \mathbb{Z}_0} X^*(\omega_k) Q(\omega_k) &= x \end{aligned}$$

When $x > x_0$, the above system no longer admits a solution, because the second equality cannot be achieved for any choice of the Lagrange multiplier y . The singular utility closes this gap, replacing the previous system by the relaxed system:

$$\begin{aligned} U' \left(\frac{\mu_a^*}{dQ}(\omega_k) \right) &= y \frac{dQ}{dP}(\omega_k) & k \in \mathbb{Z}_0 \\ \mu_s(\omega_{-\infty}) + \mu_s(\omega_{+\infty}) + \sum_{k \in \mathbb{Z}_0} \frac{d\mu_a^*}{dQ}(\omega_k) Q(\omega_k) &= x \end{aligned}$$

which contains the two additional unknowns $\mu_s(\omega_{-\infty})$ and $\mu_s(\omega_{+\infty})$. The solution to the relaxed system is obtained by choosing $y = y_0$. The value of $\mu_s(\omega_{-\infty}) + \mu_s(\omega_{+\infty})$ is thus determined from the second equation, but the two individual values $\mu_s(\omega_{-\infty})$ and $\mu_s(\omega_{+\infty})$ remain free. Indeed, since the singular utility term is $\int \varphi d\mu_s$, and $\varphi(\omega_{-\infty}) = \varphi(\omega_{+\infty})$, any measure of the form:

$$\mu^*(x) = X^*(x_0)dQ + (x - x_0)(t\delta_{\omega_{-\infty}} + (1-t)\delta_{\omega_{+\infty}}) \quad (23)$$

for any $t \in [0, 1]$ is an optimal solution.

Example 4.2 ($x_0 = \infty$, but $y_0 > 0$). Consider a bounded sequence $(\omega_n)_{n \geq 1} \subset \mathbb{R}$ decreasing to ω_∞ , and define Ω as $(\omega_n)_{n \geq 1} \cup \{\omega_\infty\}$, endowed with the Euclidean topology, under which it is Polish compact. Define P by $P(\omega_n) = p_n = e^{-n}/(e-1)$ and $P(\omega_\infty) = p_\infty = 0$. The payoff set is defined as $\mathcal{C}(x) = \{X \mid E_Q[X] \leq x\}$, where Q by $\frac{dQ}{dP}(\omega_n) = c_1/n$, where $c_1 > 1$ is a normalizing constant, and the value at ω_∞ is irrelevant. As in the previous example, a simple calculation shows that $v(y) = E_P[V(ydQ/dP)]$ is finite iff $y > 1/c_1 := y_0 > 0$. Thus, the no-bliss condition $\sup_{X \in \mathcal{C}(x)} E_P[U(X)] < \infty$ is satisfied, and $\varphi(\omega_n) = 0$ for $n \geq 1$ and $\varphi(\omega_\infty) = \frac{1}{c_1} = y_0$. $\varphi = 0$ a.s. and Assumption 1.4 holds because $\{\varphi > 0\} = \{\omega_\infty\}$ is a P -negligible set and Ω is compact.

In this model, $x_0 = -v'_+(y_0) = +\infty$ and therefore the optimal payoff X^* is classical for any $x > 0$, obtained as the unique solution to the system of equations:

$$\begin{aligned} U'(X^*(\omega_n)) &= y \frac{dQ}{dP}(\omega_n) \\ \sum_{n \geq 1} X^*(\omega_n) q_n &= x \end{aligned}$$

Example 4.3 (Dropping $\varphi = 0$ a.s.). Let Ω be as in the previous example, but modify P so that $P(\omega_\infty) > 0$. More precisely, fix $\delta \in (0, 1)$, and define P by $p_n = P(\omega_n) = (1 - \delta)e^{-n}/(e - 1)$ and $p_\infty = P(\omega_\infty) = \delta$. Likewise, define Q by $\frac{dQ}{dP}(\omega_n) = 1/n$ and $Q(\omega_\infty) = (1 - \sum_{n \geq 1} Q(\omega_n)) > 0$. Now, $v(y)$ is finite iff $y > 1$, so $y_0 = 1$. Since the continuous function:

$$g_k(\omega_n) = \begin{cases} \frac{1}{k} & \text{if } n \leq k \\ 1 + \frac{1}{k} & \text{if } +\infty \geq n > k \end{cases}$$

is in $\text{Dom}(J_V)$ for all $k \geq 1$, $\varphi(\omega_n) = 0$ for all $1 \leq n < +\infty$, while $\varphi(\omega_\infty) = 1$. For ω_∞ , the following holds:

$$W(\omega_\infty, x) = \max_{z \leq x} \left\{ U(z) + (x - z)\varphi(\omega_\infty) \frac{dQ}{dP}(\omega_\infty) \right\}$$

and consider the derivative

$$U'(x) - \varphi(\omega_\infty) \frac{dQ}{dP}(\omega_\infty)$$

If $x > x^* = (U')^{-1}(\varphi(\omega_\infty) \frac{dQ}{dP}(\omega_\infty))$, W attains its maximum at x^* , so that

$$W(\omega_\infty, x) = \begin{cases} U(x) & \text{if } x \leq x^* \\ U(x^*) + (x - x^*)\varphi(\omega_\infty) \frac{dQ}{dP}(\omega_\infty) & \text{if } x > x^* \end{cases}$$

Example 4.4 (Necessity of Assumption 1.4). The setup is the same of Example 4.1 above, only remove the points $\omega_{-\infty}, \omega_{+\infty}$. The resulting Ω is *no longer compact*, but still Polish with the Euclidean topology. As $\frac{dP}{dQ}$ is now finite everywhere and the topology is discrete, φ is identically null. However, the value function u is the same of Example 4.2, so in particular for $x > x_0$

$$u(x) = \sup_{\mathcal{C}(x)} E[U(X)] = E[U(X^*(x_0))] + y_0(x - x_0)$$

and the extra contribution cannot be given by a singular measure μ_s^* – only by a pure finitely additive μ_p^* with $\inf_{f \in \text{Dom}(J_V)} \mu_p^*(f) = y_0(x - x_0) > 0$. The maximizing sequences in $\mathcal{C}(x)$ for the value $u(x)$ are the same of Example 4.2, but this time the sequences have a weak star cluster point in $\mathcal{D}(x) \setminus \mathcal{M}_+$.

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