Shape Optimization Problems over Classes of Convex Domains

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Abstract. We consider shape optimization problems of the form

$$\min\left\{\int_{\partial A} f(x,\nu(x))\,\mathrm{d}\mathcal{H}^{n-1} : A\in\mathcal{A}\right\}$$

where f is any continuous function and the class $\mathcal{A}$ of admissible domains is made of convex sets. We prove the existence of an optimal solution provided the domains satisfy some suitable constraints.

1. Introduction

The problem of finding the shape with least resistance for a body moving in a fluid has a long history: it was first posed in 1685 by Newton in his *Principia Mathematica*, at the early stages of Calculus of Variations.

Newton's model was very simple: he supposed a fluid is made of many particles of equal mass, and the resistance is given by the elastic interactions between the particles and the front part of the body. Effects due to viscosity and turbulence are thus neglected by this model, which nevertheless gives good results in some interesting cases, as for bodies moving in an ideal gas with high Mach number, for rarefied gases and low speed, and in the case of slender bodies (see for instance Funk [4], and Miele [7] for several interesting applications).

From the mathematical point of view, Newton's model provides a resistance functional which has many unpleasant properties: in fact it is neither coercive nor convex, so the classical direct method does not provide the existence of a minimum, unless we put additional constraints on the class of admissible bodies.

In the radial case (i.e. considering only shapes with circular section and prescribed cylindrical symmetry), Newton himself guessed the solution, which in this case is unique, and observed some qualitative properties that hold in general, although the first proof of existence was given by Kneser [5] in 1902.

In recent years new interest has raised about Newton's problem: Marcellini [6] found a new proof of existence in the radial case, which provides also the concavity of the minimizer and the validity of Euler's equation. The nonparametric problem without symmetry (i.e. for profiles which are graphs of Cartesian functions defined on a fixed open set) was studied by Buttazzo & Kawohl [3] and Buttazzo, Ferone & Kawohl [2], who established the existence of minimizers in the classes of concave and superharmonic Cartesian profiles. Moreover, Brock, Ferone & Kawohl [1] proved that, even in the case of prescribed circular cross section, the minimizers do not fulfill any cylindrical symmetry.

In order to address the parametric problem, here we present a different approach, which allows us to write Newton's functional independently of the local representation. Indeed, the Newtonian resistance of a n-dimensional body E can also be written in the form

$$F(E) = \int_{\partial E} f(x, \nu(x)) \, \mathrm{d}\mathcal{H}^{n-1} ,$$

with $f(x,\nu) = ((a \cdot \nu)^+)^3$, being a the direction of motion.

More generally, we consider cost functionals of the above form with a continuous function f and we prove an existence result for minimizers of F in the classes

$$C_{K,Q} = \{E \text{ convex subset of } \mathbb{R}^n\colon K \subset E \subset Q\}$$
$$\mathcal{C}_{V,Q} = \{E \text{ convex subset of } \mathbb{R}^n\colon E \subset Q,\ |E| \geq V\} .$$

Finally, in the last section we let also the "cross section" vary in a suitable class, and we show that the existence of a minimum can be obtained under some constraints.

2. Preliminaries and Main Result

Let $E \subset \mathbb{R}^n$ be a smooth body, whose profile is defined by the graph of a function $u : \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^{n-1}$. Following Newton's assumptions, suppose that the fluid is made of several particles moving vertically (i.e. orthogonally to Ω) with the same velocity. Assume also that each particle hits E *elastically* and *only once*, and that the body is constrained on a vertical guide, so that the horizontal components of the shocks are neglected.

As it is easily seen in figure 1, the momentum that E receives is proportional to $\sin^2\theta$, where θ is the incidence angle of the particle with respect to the tangent plane to E in the hitpoint. Expressing $\sin^2\theta$ as $\frac{1}{1+|\nabla u|^2}$ and integrating over Ω, we obtain the following expression for the resistance, up to multiplication by a constant depending on the fluid's velocity and density:

$$F(u) = \int_{\Omega} \frac{1}{1+|\nabla u|^2} \, \mathrm{d}x . \tag{1}$$

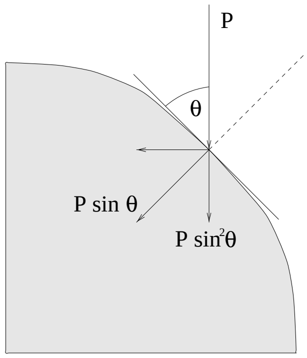

FIGURE 1. Newton's model for resistance

By Rademacher's theorem, this functional is well defined on every locally Lipschitz function, but the assumption that each particle hits the body only once implies some complicated constraints on the geometry of the body (see [2] for more details). However, this single shock property is always satisfied by convex profiles, so the choice of working with concave functions looks quite convenient. In fact, in [2] it is proved that (1) has a minimum in the class C_M, defined as follows:

$$C_M = \{u \text{ concave on } \Omega : 0 \leq u \leq M\} \ .$$

A different approach to the problem consists of writing (1) only in terms of the set E, without dependence on its Cartesian representation. Denoting by $\nu(x)$ the exterior normal to E in x, we have that

$$\nu = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}} \qquad \text{and} \qquad \frac{\mathrm{d}(\mathcal{H}^{n-1} \llcorner \operatorname{graf} u)}{\mathrm{d}(L^{n-1} \llcorner \Omega)} = \sqrt{1+|\nabla u|^2} = \frac{1}{\nu_n} \ ,$$

where ν_n denotes the n-th component of ν. Thus, changing variable with respect to $x \mapsto (x, u(x))$, the resistance can be written in the form

$$F(u) = \int_{\Omega} \nu_n^2 \, \mathrm{d}x = \int_{\operatorname{graf} u} \nu_n^3 \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial E} (\nu_n^+)^3 \, \mathrm{d}\mathcal{H}^{n-1} \ ,$$

where in the last equality the positive part is used in order to neglect the integral on $\partial E \setminus \operatorname{graf} u$. Note that since convex sets are locally Lipschitz, the exterior normal ν is defined for a.e. point of the boundary ∂E. Thus the previous expression makes sense for each convex set E, and we can define the new functional as

$$F(E) = \int_{\partial E} (\nu_n^+)^3 \, \mathrm{d}\mathcal{H}^{n-1} \ . \tag{2}$$

In general, we consider minimum problems of the form

$$\min \big\{ F(E) : \ E \in \mathcal{A} \big\}$$

where the functional F is given by

$$F(E) = \int_{\partial E} f(x, \nu(x))\ \mathrm{d}\mathcal{H}^{n-1}$$

with f continuous, and the class $\mathcal{A}$ of admissible sets is a class of convex sets. More precisely we consider the classes

$$C_{K,Q} = \{E \text{ convex subset of } \mathbb{R}^n\text{: } K \subset E \subset Q\}$$
$$\mathcal{C}_{V,Q} = \{E \text{ convex subset of } \mathbb{R}^n\text{: } E \subset Q,\ |E| \geq V\}\ ,$$

where K and Q are two compact subsets of $\mathbb{R}^n$ and V is a positive number. Denoting by S^{n-1} the $(n-1)$-dimensional sphere in $\mathbb{R}^n$, the main result of this paper can be stated as follows:

Theorem 2.1 *Let $f : \mathbb{R}^n \times S^{n-1} \mapsto \mathbb{R}$ be a bounded continuous function. For each convex $E \subset \mathbb{R}^n$ we set*

$$F(E) = \int_{\partial E} f(x, \nu(x))\ \mathrm{d}\mathcal{H}^{n-1}\ .$$

Then the minimum problems

$$\min_{E \in C_{K,Q}} F(E) \qquad \textit{and} \qquad \min_{E \in \mathcal{C}_{V,Q}} F(E)$$

admit at least one solution.

The proof of the theorem requires some tools of geometric measure theory that we recall briefly (see for instance Ziemer [10] for a complete discussion on the topic):

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then each bounded subset of the space $BV(\Omega)$ is relatively compact with respect to the strong $L^1_{\mathrm{loc}}(\Omega)$ convergence.*

Definition 2.3. For each Borel measure μ on $\mathbb{R}^n$ with values in $\mathbb{R}^n$ and each Borel subset B of $\mathbb{R}^n$ we define the *variation* of μ on B as

$$|\mu|(B) = \sup_{\{B_n\} \in \mathfrak{P}(B)} \sum_n |\mu(B_n)|\ ,$$

where $\mathfrak{P}(B)$ is the class of countable partitions of B. We denote by $\mathcal{M}$ the class of all measures μ such that $|\mu|(\mathbb{R}^n) < +\infty$, and for each $\mu \in \mathcal{M}$ we set

$$\|\mu\| = |\mu|(\mathbb{R}^n)\ .$$

Definition 2.4. Let (μ_h) be a sequence of measures in $\mathcal{M}$. We say that (μ_h) *converges in variation* to μ if (μ_h) tends to μ in the weak * convergence of measures and $\lim_{h\to\infty} \|\mu_h\| = \|\mu\|$.

The following theorem was first proved by Reshetnyak [8], and establishes the continuity of a large class of functionals with respect to the convergence in variation:

Theorem 2.5 (Reshetnyak [8]) *Let $f : \mathbb{R}^n \times S^{n-1} \to \mathbb{R}$ a bounded continuous function. Then the functional $F : \mathcal{M} \to \mathbb{R}$ defined as*

$$F(\mu) = \int_{\mathbb{R}^n} f(x, \nu_\mu) \, \mathrm{d}|\mu|$$

is continuous with respect to the convergence in variation. Here ν_μ denotes the Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}$.

For convex sets the following representation of $\mathcal{H}^{n-1} \llcorner \partial E$ holds, where χ_E denotes the characteristic function of E:

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \ . \end{cases}$$

Theorem 2.6 *Let E be a convex set. Then the measures $|D\chi_E|$ and $\mathcal{H}^{n-1} \llcorner \partial E$ coincide.*

3. Proof of the main result

The proof of Theorem 2.1 is based on the direct method of the calculus of variations: first we prove a strong compactness property for the class of convex sets, then we restate the problem in terms of functionals depending on vector measures, where Reshetnyak's theorem applies.

Proof of Theorem 2.1. Using Theorem 2.6, the functional F can be written in the form

$$F(E) = \int_{\partial E} f(x, \nu) \, \mathrm{d}\mathcal{H}^{n-1} = \int_{Q} f(x, \nu) \, \mathrm{d}|D\chi_E| \ ,$$

where the vector measure $D\chi_E$ is the derivative of the characteristic function of E in the sense of Schwartz distributions, and $|D\chi_E|$ is the variation measure associated to $D\chi_E$. By the Radon-Nikodym theorem, we have the equality

$$\nu = -\frac{\mathrm{d}D\chi_E}{\mathrm{d}|D\chi_E|} \ ,$$

so we can think of F as a functional defined on the following classes of measures:

$$\begin{aligned} M_{K,Q} =& \big\{ \mu \in \mathcal{M} : \mu = D\chi_E, \ E \in C_{K,Q} \big\} \\ \mathcal{M}_{V,Q} =& \big\{ \mu \in \mathcal{M} : \mu = D\chi_E, \ E \in \mathcal{C}_{V,Q} \big\} \ . \end{aligned}$$

Lemma 3.1 below provides the compactness of the classes $M_{K,Q}$ and $\mathcal{M}_{V,Q}$, while Theorem 2.5 gives the continuity of F, so existence follows by the direct method of the calculus of variations. □

Lemma 3.1 *The classes $M_{K,Q}$ and $\mathcal{M}_{V,Q}$ are compact with respect to the convergence in variation.*

The proof requires some lemmas:

Lemma 3.2 (see 2.4 of [2]) *Let $A, B \subset \mathbb{R}^n$ be two n-dimensional convex sets such that $A \subset B$. Then $H^{n-1}(\partial A) \leq H^{n-1}(\partial B)$ and equality holds iff $A = B$.*

Proof. Let $P : \partial B \to \partial A$ be the projection on the convex $\overline{A}$, which maps each point of ∂B in the point of ∂A of least distance from A. It is well known that P is Lipschitz with Lipschitz constant 1. By the properties of Hausdorff measures, (see for instance Rogers [9], Theorem 29), we get the inequality

$$\mathcal{H}^{n-1}(\partial A) = \mathcal{H}^{n-1}(P(\partial B)) \leq \mathcal{H}^{n-1}(\partial B) \ .$$

Finally, if $\mathcal{H}^{n-1}(\partial A) = \mathcal{H}^{n-1}(\partial B)$ and by contradiction $A \neq B$, we could find a hyperplane S tangent to A such that, denoting by S^+ the half space bounded by S and containing A, it is

$$B \setminus S^+ \neq \emptyset \ .$$

It is easy to see that $B \setminus S$ contains an open set, so that

$$\begin{aligned} (3)\quad & \mathcal{H}^{n-1}(\partial A) \leq \mathcal{H}^{n-1}\left(\partial(B \cap S^+)\right) = \\ & = \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}\left(\partial B \setminus S^+\right) + \mathcal{H}^{n-1}(S \cap B) < \mathcal{H}^{n-1}(\partial B) \ , \end{aligned}$$

which contradicts the assumption $\mathcal{H}^{n-1}(\partial A) = \mathcal{H}^{n-1}(\partial B)$. □

Lemma 3.3 (see 4.4 of [2]) *Let $E_h, E \subset \mathbb{R}^n$ be bounded convex subsets of $\mathbb{R}^n$ with $E_h \to E$ in $L^1(\Omega)$. Then $\mathcal{H}^{n-1}(\partial E_h) \to \mathcal{H}^{n-1}(\partial E)$.*

Proof. The proof is trivial if E has dimension smaller than n, so it is sufficient to consider the case $\dim_{\mathcal{H}} E = n$. As E_h converges to E in $L^1(\Omega)$ it follows that

$$\forall \varepsilon > 0 \quad \exists h_\varepsilon : h > h_\varepsilon \quad \Longrightarrow \quad E_h \subset E + B(0,\varepsilon) \ .$$

Therefore, by Lemma 3.2, we obtain for $h > h_\varepsilon$,

$$\mathcal{H}^{n-1}(\partial E_h) \leq \mathcal{H}^{n-1}(\partial(E + B(0,\varepsilon))) \ ,$$

so that

$$\limsup_{h\to\infty} \mathcal{H}^{n-1}(\partial E_h) \leq \limsup_{\varepsilon\to 0^+} \mathcal{H}^{n-1}(\partial(E + B(0,\varepsilon))) = \mathcal{H}^{n-1}(\partial E) \ ,$$

But by the L^1 lower semicontinuity of the perimeter,

$$\liminf_{h\to\infty} \mathcal{H}^{n-1}(\partial E_h) \geq \mathcal{H}^{n-1}(\partial E) \ ,$$

and the proof is complete. □

Proof of Lemma 3.1. We set

$$C_Q = \{E \text{ convex} : E \subset Q\} \ ,$$

and the corresponding class of measures

$$\mathcal{M}_Q = \{\mu \in \mathcal{M} : \mu = D\chi_E,\ E \in C_Q\} \ ,$$

Let E be an element of C_Q. We have, by Lemma 3.2,

$$|D\chi_E|(Q) = \mathcal{H}^{n-1}(\partial E) \leq \mathcal{H}^{n-1}(\partial Q) \ ,$$

and therefore the class C_Q is a bounded subset of BV, while $\mathcal{M}_Q$ is bounded in $\mathcal{M}$. For each sequence E_h of convex sets contained in Q, by the Banach-Alaoglu theorem we can assume up to a subsequence that $D\chi_{E_h}$ tends to $D\chi_E$ in the weak * topology. Moreover, by the compactness theorem for BV functions, we can assume that $\chi_{E_h} \to \chi_E$ in L^1 strongly. By Lemma 3.3

$$\lim_{h\to\infty} |D\chi_{E_h}|(\Omega) = \lim_{h\to\infty} \mathcal{H}^{n-1}(\partial E_h) = \mathcal{H}^{n-1}(\partial E) = |D\chi_E|(\Omega) \ ,$$

so the compactness with respect to the convergence in variation for the class $\mathcal{M}_Q$ follows. Since the classes $M_{K,Q}$ and $\mathcal{M}_{V,Q}$ are clearly closed with respect to the convergence in variation, the proof is achieved. □

Remark 3.4. In Theorem 2.1 the assumption of convexity for the sets in classes $C_{K,Q}$ and $\mathcal{C}_{V,Q}$ cannot be easily weakened. Indeed, consider for instance the sequence (E_h) given by:

$$E_h = \left\{(x,y) \in \mathbb{R}^2 : x \in [0,1], 0 \leq y \leq 1 + \frac{\sin hx}{h}\right\} \ .$$

The sets E_h are equilipschitz domains, converge in L^1 and uniformly (in the sense of Lemma 4.2 below) to $E = [0,1] \times [0,1]$. Nevertheless, taking the Newton functional (2) we have that the lower semicontinuity inequality does not hold, i.e. $F(E) > \lim_{h\to\infty} F(E_h)$. In the Cartesian case this is due to the lack of convexity of (1) with respect to the gradient.

4. Section Optimization

In the nonparametric case of Newton's problem we seek the shape with least resistance among those having prescribed cross section Ω. We can also let Ω vary in a suitable class, so that also the optimal section becomes an unknown.

More generally, we consider the class

$$S_{A,K,Q} = \{E \text{ convex closed subset of } \mathbb{R}^n\text{: } K \subset E \subset Q, \mathcal{H}^{n-1}(E \cap \pi) \geq A\}$$

where π is a given hyperplane in $\mathbb{R}^n$, while K and Q are two compact subsets of $\mathbb{R}^n$ with positive measure, and A is a positive number (which is assumed smaller than $\mathcal{H}^{n-1}(Q \cap \pi)$ and bigger than $\mathcal{H}^{n-1}(K \cap \pi)$).

The following result holds:

Theorem 4.1 *Let F be as in Theorem 2.1. Then the minimum problem*

$$\min_{E \in S_{A,K,Q}} F(E)$$

admits at least one solution.

The proof requires some lemmas:

Lemma 4.2 *Let $E_h, E \subset \mathbb{R}^n$ be bounded convex subsets of $\mathbb{R}^n$ with positive measure and such that $E_h \to E$ in L^1. Then*

$$\forall \varepsilon > 0 \; \exists h_\varepsilon : h > h_\varepsilon \Longrightarrow E_h \triangle E \subset \{x : \operatorname{dist}(x, \partial E) \leq \varepsilon\} \; .$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Proof. Suppose $\chi_{E_h}(x) \neq \chi_E(x)$ and let $d = \operatorname{dist}(x, \partial E) > 0$.

Consider first the case $x \in \overset{\circ}{E}$, $x \notin E_h$. By the convexity of E we can find a closed halfspace S such that $x \in \partial S$ and $E_h \cap S = \emptyset$. Let B_d be the open ball centered in x with radius d. Then we have that

$$\|\chi_{E_h} - \chi_E\|_{L^1} \geq |E \setminus E_h| \geq |S \cap B_d| = \frac{1}{2}\omega_n d^n$$

where $\omega_n = \mathcal{H}^n(B_1)$. This inequality implies with L^1 convergence that $d \to 0$ as $h \to \infty$.

Now consider the case $x \in \mathbb{R}^n \setminus \overline{E}$, $x \in E_h$. For the previous case we can assume that for each ε there exists $\tilde{h}$ such that if $h > \tilde{h}$ then $K_\varepsilon = \{x \in E : \operatorname{dist}(x, \partial E) \geq \varepsilon\} \subset E_h$. Denote by z the point of ∂E of minimal distance from x, by π the hyperplane orthogonal to the segment $\overline{x\,z}$ and passing by an interior point y of K_ε. Analogously, π' will be the hyperplane parallel to π and passing by z. Let C be the cone with vertex x and basis $K_\varepsilon \cap \pi$, and C' the cone with vertex x

and basis $C \cap \pi'$. Then we have:

$$|C'| = \frac{1}{n}\frac{\mathcal{H}^{n-1}(K_\varepsilon \cap \pi)}{\mathrm{dist}(x,y)^{n-1}}\; d^n$$

and thus

$$\|\chi_{E_h} - \chi_E\|_{L^1} \geq |E_h \setminus E| \geq |C'|$$

which implies that $d \to 0$ as $h \to \infty$. This concludes the proof. □

Lemma 4.3 *Let π be a hyperplane, and $E_h, E \subset \mathbb{R}^n$ be bounded convex closed subsets of $\mathbb{R}^n$ with positive measure, such that $E_h \to E$ in $L^1(\Omega)$. Then $\limsup_{h\to\infty} \mathcal{H}^{n-1}(E_h \cap \pi) \leq \mathcal{H}^{n-1}(E \cap \pi)$.*

Proof. It is sufficient to prove that

$$\forall x \in \mathbb{R}^n \quad \limsup_{h\to\infty} \chi_{E_h\cap\pi}(x) \leq \chi_{E\cap\pi}(x)\ .$$

Let $x_{h_k} \in E_{h_k} \cap \pi$ such that $x_{h_k} \to x$ and suppose, by contradiction, that $x \notin E \cap \pi$. Then $\mathrm{dist}(x, \partial E) > 0$, and by Lemma 4.2 we know that $\chi_{E_{h_k}}$ vanishes eventually in a neighborhood of x. But this clearly contradicts the assumption that $x_{h_k} \in E_{h_k}$. □

Proof of Theorem 4.1. From Lemma 3.1 we know that the class of measures

$$M_{K,Q} = \{\mu \in \mathcal{M} : \mu = D\chi_E,\ E \in C_{K,Q}\}$$

is compact with respect to the convergence in variation, so it is sufficient to prove that if $E_h \to E$ in L^1, then $\limsup_{h\to\infty} \mathcal{H}^{n-1}(E_h \cap \pi) \leq \mathcal{H}^{n-1}(E \cap \pi)$, which is given by Lemma 4.3. □

Remark 4.4. In Theorem 2.1, if f does not depend on the variable x, we have that the functional F is homogeneous of degree $n-1$ with respect to homotheties on the depending set. This implies that the boundary of the minimizer E in the class $C_{K,Q}$ must touch either ∂K or ∂Q. In the case of classes $\mathcal{C}_{V,Q}$ and $S_{A,K,Q}$ the situation seems more complicated, and we did not find any general classification result.

Remark 4.5. By repeating the arguments used in the proofs, one can obtain the analogous results of Theorem 2.1 and 4.1 for the classes

$$\tilde{\mathcal{C}}_{V,Q} = \{E \text{ convex subset of } \mathbb{R}^n\text{: } E \subset Q,\ |E| = V\}$$
$$\tilde{S}_{A,K,Q} = \{E \text{ convex closed subset of } \mathbb{R}^n\text{: } K \subset E \subset Q, \mathcal{H}^{n-1}(E \cap \pi) = A\}$$

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