MEAN-VARIANCE HEDGING FOR STOCHASTIC VOLATILITY MODELS

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Abstract. In this paper we discuss the tractability of stochastic volatility models for pricing and hedging options with the mean-variance hedging approach. We characterize the variance-optimal measure as the solution of an equation between Doleans exponentials: explicit examples include both models where volatility solves a diffusion equation, and models where it follows a jump process.

We further discuss the closedness of the space of strategies.

Introduction

The mean-variance hedging approach to pricing and hedging contingent claims was introduced (in the martingale case) by Föllmer/Sondermann (1986) : subsequent extensions to the general semimartingale case were done by Duffie/Richardson (1991) , Schweizer (1992, 1996) , Schäl (1994) , Gouriéroux et al. (1998) , Pham et al. (1997) and Rheinländer/Schweizer (1997). The paper of Schweizer (1999) contains a general overview of the subject, and a complete bibliography.

The aim of this paper is to analyse the mean-variance hedging criterion in stochastic volatility models : we develop a general framework (introduced by Föllmer/Schweizer (1991)) where a stochastic volatility model is seen as a model with incomplete information.

This model would be complete with respect to some larger filtration (usually including all information on past and future volatility), but not under the filtration available to the hedging agent (who usually observes only the asset price history). This framework is general enough to include both the diffusion models (such as Hull-White, Heston, Stein and Stein, only to mention a few), and less common models where volatility jumps.

We begin our analysis with a characterization of the set of equivalent martingale measures in terms of *Doléans exponentials*: this provides a one-one correspondence between equivalent martingale measures and a class of predictable processes. Exploiting results of Schweizer (1996) and Delbaen/Schachermayer (1996) , we then identify the variance-optimal martingale measure as the solution of an equation involving exponential martingales.

Our results are illustrated by several examples: the detailed analysis of all these examples can be found in Biagini/Guasoni (1999).

In the case of diffusion stochastic volatility models we recover some results of Laurent/Pham (1999) with a different method: in fact, while they use dynamic programming techniques, we essentially focus on stochastic integration. Also the

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recent paper by Heath et al. (1999) contains a detailed analysis of the mean-variance hedging criterion (compared to the locally risk-minimizing criterion) in stochastic volatility models.

In order to keep notations simple, we only consider one-dimensional models; however our results can easily be extended to the multidimensional case.

1. Statement of the problem

For all definitions on stochastic integration and martingale representation, we refer to Protter (1990) and Dellacherie/Meyer (1982) (in particular, all filtrations are supposed to satisfy the so-called *usual hypothesis*).

We have two complete filtered probability spaces denoted by $(\Omega, \mathcal{F}^W, \mathcal{F}_t^W, P^W)$ and $(E, \mathcal{E}, \mathcal{E}_t, P^E)$. We assume that W_t is a standard Brownian Motion on $\Omega = C([0,T], \mathbb{R})$, P^W is the standard Wiener measure, $\mathcal{F}^W = \mathcal{F}_T^W$, and $\mathcal{F}_t^W$ is the P^W-augmentation of the filtration generated by W.

We have two assets: the risky asset S_t, and the riskless asset $B_t = \exp\left(\int_0^t r_s ds\right)$, where r_t is a deterministic function. The risky asset is represented by a process $S_t(w, \eta)$ on the product space $\Omega \times E$, whose dynamics is given by the following equation:

$$\begin{cases} dS(t,\omega,\eta) = S(t,\omega,\eta)\big(\mu(t,\omega,\eta)dt + \sigma(t,\omega,\eta)dW_t(\omega)\big) \\ S(0) = S_0 \end{cases}$$

We shall make the following assumptions:

i) on the space E we have a (possibly d-dimensional) martingale M which has the predicable representation property with respect to the filtration $(\mathcal{E}_t)_{t\in[0,T]}$.
ii) the information available at time t is given by the filtration $\mathcal{F}_t^W \otimes \mathcal{E}_t$.
iii) the probability P on $\Omega \otimes E$ is the product probability $P^W \otimes P^E$.

Remark 1.1. In many applications, the most natural filtration available to the agent is the one generated by S: let us see how $ii)$ translates in this case. If σ has a right-continuous version, it is $\mathcal{F}_t^S$-adapted: in fact we recall that (see Föllmer/Schweizer (1991) page 410)

$$\langle S\rangle_t = \int_0^t \sigma_s^2 S_s^2 ds = \lim_{\sup_i |t_{i+1}-t_i|\to 0} \sum (S_{t_{i+1}} - S_{t_i})^2$$

is $\mathcal{F}_t^S$-adapted. If $\mu(t,\omega,\eta)$ is also $\mathcal{F}_t^S$-adapted, it is easy to see that the filtration generated by S coincides with the one generated by (W, μ, σ). Therefore the assumption $ii)$ boils down to

$$\mathcal{F}_t^{W,\mu,\sigma} = \mathcal{F}_t^W \otimes \mathcal{E}_t$$

Remark 1.2. Since the technicalities involved in the definition above may hide the idea of incomplete information, we provide a simple explanation. This market would be complete if the agent had access to the (larger) filtration $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{E}$, which contains at any time all the information on past and future drift and volatility. As pointed out by Föllmer/Schweizer (1991) , this is a consequence of the fact that all

$\widetilde{\mathcal{F}}_t$-martingales can be written in the form

$$N_t(\omega,\eta) = N_0(\eta) + \int_0^t H_s(\omega,\eta)dW_s(\omega)$$

for some $\widetilde{\mathcal{F}}$-predictable process H. This result is an exercise on stochastic integration: we provide a proof in the appendix (Proposition 5.1), for the sake of completeness.

The discounted value of the risky asset follows the equation:

$$\begin{cases} dX_t = X_t\big((\mu_t - r_t)dt + \sigma_t dW_t\big) \\ X_0 = S_0 \end{cases}$$

We assume μ and σ are such that $X_t \in L^2(P)$ for all $t \in [0,T]$, and denote by $\lambda_t(\omega,\eta) = \dfrac{\mu(t,\omega,\eta) - r(t)}{\sigma(t,\omega,\eta)}$ the so-called *market price of risk*.

Example 1.3. This example was introduced by Harrison/Pliska (1981) , and investigated later by Föllmer/Schweizer ((1991) , page 142).

μ_t and σ_t are constant until a fixed time t_0, then they jump simultaneously, the pair (μ,σ) having two possible outcomes. In other words

$$\begin{cases} \mu_t(\eta) = 1_{\{t<t_0\}}\mu + 1_{\{t\geq t_0\}}\mu_\eta \\ \sigma_t(\eta) = 1_{\{t<t_0\}}\sigma + 1_{\{t\geq t_0\}}\sigma_\eta \end{cases}$$

where $E = \{0,1\}$, $\mathcal{E}_t = \{\emptyset, E\}$ for $t < t_0$ and $\mathcal{E}_t = \mathcal{P}(E)$ for $t \geq t_0$. A fundamental martingale is given by $M_t = 1_{\{t\geq t_0\}}(1_{\{\eta=1\}} - p)$, where $p = P(\eta = 1)$.

This example was generalized by Föllmer/Leukert (1999) , where the values of μ_t and σ_t after the jump time t_0 have a continuous distribution: in this case $E = \mathbb{R}$ and the martingale M has to be replaced by a random measure (see Biagini/Guasoni (1999) for details).

Example 1.4. The previous example can be extended in several ways: we consider in particular a model proposed in discrete time fashion in RiskMetrics Monitor (see Zangari (1996)) as an improvement of the standard lognormal model for calculating Value at Risk. More precisely, we have multiple independet jumps at fixed equispaced time intervals. We can set $E = \{0,1\}^n$ and, denoting $\eta = \{a_1,\ldots,a_n\}$, $\mathcal{E}_t$ is equal to the parts of $\{a_i\}_{t_i\leq t}$ (where $t_i = i\frac{T}{n+1}$). One obtains the following dynamics:

$$\begin{cases} \mu_t(\eta) = 1_{\{t<t_1\}}\mu + \sum_i 1_{\{t_i\leq t<t_{i+1}\}}\mu_{a_i} + 1_{\{t\geq t_n\}}\mu_{a_n} \\ \sigma_t(\eta) = 1_{\{t<t_1\}}\sigma + \sum_i 1_{\{t_i\leq t<t_{i+1}\}}\sigma_{a_i} + 1_{\{t\geq t_n\}}\sigma_{a_n} \end{cases}$$

Since in this model η is binomially distributed (in fact the numbers a_i are independent and $P(a_i = 1) = p$) , it is evident the existence of a martingale with the representation property.

Example 1.5. This example was studied in detail by Biagini/Guasoni (1999). We have

$$\mu_t(\eta) = 1_{\{t<\tau\}}\mu_1 + 1_{\{t\geq\tau\}}\mu_2$$
$$\sigma_t(\eta) = 1_{\{t<\tau\}}\sigma_1 + 1_{\{t\geq\tau\}}\sigma_2$$

where τ is a stopping time whose law restricted to $[0,T)$ has a density f (and $P(\tau = T) = 1 - \int_0^T f(s)ds$).

In this case, $E = [0,T]$, $\mathcal{E}_t = \mathcal{B}([0,t]) \cup (t,T]$, and a fundamental martingale can be found in the form $M_t = 1_{\{t \geq \tau\}} - a(t \wedge \tau)$, where $a(.)$ is an increasing function which can be explicitely determined in terms of f.

Example 1.6. The previous example can be generalized in the following way: after the jump time τ, μ and σ have a general probability distribution independent from τ. The space E is, in this case, $[0,T] \times \mathbb{R}$ and the martingale M is replaced by the random measure $\left(\nu - \nu^p\right)$, where ν^p is the compensator of the random measure $\nu(\eta, dt, dx) = \epsilon_{(\tau(\eta), \alpha(\eta))}(dt, dx)$ and $\alpha(\eta) = \lambda^2(\eta) - \lambda_1^2$.

Example 1.7. A number of diffusion stochastic volatility models have been proposed in the literature, most of them being particular cases of the following

$$\begin{cases} dX_t = \sigma(t, X_t, v_t) X_t \big(\lambda(t, X_t, v_t)dt + dW_t^1\big) \\ dv_t = \alpha(t, X_t, v_t)dt + \beta(t, X_t, v_t)dW_t^1 + \gamma(t, X_t, v_t)dW_t^2 \end{cases}$$

where W^1 and W^2 are two independent Brownian Motions.

We set $E = C([0,1], \mathbb{R})$, and $\mathcal{E}_t$ is the augmentation of the filtration generated by W_t^2 : the natural choice for a martingale with the representation property on E is clearly W^2.

In the general framework described above, an agent wishes to hedge a certain European option H expiring at a fixed time T : his goal is to minimize the risk, defined as the variance of the tracking error at expiration. Therefore we look for a solution to the minimum problem

$$\min_{\substack{c \in \mathbb{R} \\ \theta \in \Theta}} E\left[(H - c - G_T(\theta))^2 \right] \tag{1}$$

where

$$G_t(\theta) = \int_0^t \theta_s dX_s \qquad \text{and} \qquad \Theta = \left\{ \theta \in L(X),\ G_t(\theta) \in \mathcal{S}^2(P) \right\}$$

Here $L(X)$ denotes the space of X-integrable $\mathcal{F}_t$-predictable processes, and $\mathcal{S}^2$ the space of semimartingales Y decomposable as $Y = Y_0 + M + A$, where M is a square-integrable martingale, and A is a process of square-integrable variation.

Definition 1.8. We define the following spaces of signed martingale measures

$$\begin{aligned} \mathcal{M}_s &= \{Q \ll P\ : X_t \text{ is a } Q\text{-local martingale}\} \\ \mathcal{M}_e &= \{Q \in \mathcal{M}_s, Q \sim P\} \\ \mathcal{M}_s^2 &= \left\{ Q \in \mathcal{M}_s, \frac{dQ}{dP} \in L^2(P) \right\} \\ \mathcal{M}_e^2 &= \mathcal{M}_e \cap \mathcal{M}_s^2 \end{aligned}$$

If Q is a signed probability with density Z with respect to P, by definition X_t is a Q-martingale if $X_t Z_t$ is a P-martingale, where $Z_t = E\left[Z | \mathcal{F} \right]$.

The existence of a minimizer for (1) was shown for any $H \in L^2(P)$ independently by

Gouriéroux et al (1998) and Rheinländer/Schweizer (1997) under the two standing assumptions (which need to be checked for each particular model):

i) $\mathcal{M}_e^2 \neq \emptyset$;

ii) $G_T(\Theta)$ is closed.

While *i)* is equivalent to a no-arbitrage condition (see Delbaen/Schachermayer (1996)) and holds for very general models, *ii)* often fails even for models commonly used in practice. However we shall return to this issue later.

If (1) has a solution, the optimal value for c can be written as

$$c = \widetilde{E}[H]$$

where $\widetilde{E}$ denotes the expectation under a new signed measure $\widetilde{P}$, the so-called variance-optimal martingale measure. By definition, $\widetilde{P}$ is the element of minimal norm in $\mathcal{M}_s^2$ (which evidently exists as soon as $\mathcal{M}_s^2 \neq \emptyset$): see for instance Schweizer (1996) for further details.

Our first step towards an explicit formula for $\dfrac{d\widetilde{P}}{dP}$ is the characterization of the set $\mathcal{M}_e^2$ of the square-integrable equivalent martingale measures. We start by recalling the following:

Definition 1.9. The *Doléans exponential* $\mathcal{E}(Z)$ of a semimartingale Z is defined as

$$\mathcal{E}(Z)_t = \exp\left(Z_t - \frac{1}{2}\langle Z^c\rangle_t\right)\prod_{s\leq t}(1+\Delta Z_s)\exp(-\Delta Z_s)$$

where Z^c denotes the continous part of Z, while $\Delta Z_s = Z_s - Z_{s^-}$.

We prove now the following lemma.

Lemma 1.10. *Let Z_t be a local martingale with $Z_0 = 1$. The following conditions are equivalent:*

i) Z_tX_t is a local martingale

ii) $Z_t = \mathcal{E}\left(-\int_0^\cdot \lambda_s dX_s\right)_t (1 + \int_0^t k_s dM_s)$ for some predictable process k_s such that the stochastic integral $\int_0^t k_s dM_s$ is a local martingale.

Proof. We recall that the pair (W, M) has the predictable representation property (see Proposition 5.2). Therefore

$$Z_t = 1 + \int_0^t h_s dW_s + \int_0^t k_s dM_s$$

By Itô's formula, we have

$$d(Z_tX_t) = \big[Z_{t-}(\mu_t - r_t) + h_t\sigma_t\big]X_t dt + \big[Z_{t-}\sigma_t X_t + h_t X_t\big]dW_t + k_t X_t dM_t$$

The process (Z_tX_t) is a local martingale if and only if $h_t = -\dfrac{\mu_t - r_t}{\sigma_t}Z_{t-}$. More precisely, if $\lambda_t = \dfrac{\mu_t - r_t}{\sigma_t}$, Z_t satisfies the following stochastic differential equation:

$$dZ_t = -\lambda_t Z_{t-} dW_t + k_t dM_t$$

which has a unique solution (see Protter (1990) for details). It can easily be verified that $Z_t = \mathcal{E}\left(-\int_0^\cdot \lambda_s dX_s\right)_t (1+\int_0^t k_s dM_s)$ is the solution of the above equation. $\square$

If Z_T is strictly positive, then $N_t = 1+\int_0^t k_s dM_s$ can be written as the *Doléans exponential* $N_t = \mathcal{E}\left(-\int_0^\cdot \frac{k_s}{N_{s-}} dM_s\right)_t$.

An immediate consequence of the above lemma is the characterization of $\mathcal{M}_s^2$ and $\mathcal{M}_e^2$.

Proposition 1.11. 1. *For every* $Q \in \mathcal{M}_s^2$

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^\cdot \lambda_t(\omega,\eta) dW_t\right)_T \left(c+\int_0^T k_t dM_t\right)$$

where k_t *is a process such that the above expression is square integrable.*

2. *For every* $Q \in \mathcal{M}_e^2$

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^\cdot \lambda_t(\omega,\eta) dW_t\right)_T \mathcal{E}\left(\int_0^\cdot k_t(\omega,\eta) dM_t\right)_T$$

with k_t *such that* $k_t \cdot \Delta M_t > -1$ *and* $\mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s + k_s dM_s\right)_t$ *is a square-integrable martingale.*

Recall that $\mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s + k_s dM_s\right)_t = \mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s\right)_t \mathcal{E}\left(\int_0^\cdot k_s dM_s\right)_t$ since $[W,M]=0$ (see Protter (1990) pag. 79). Condition $k_t \cdot \Delta M_t > -1$ guarantees the positivity of $\mathcal{E}\left(\int_0^\cdot k_t dM_t\right)_T$.

Remark 1.12. A similar characterization holds for the probabilities $Q \ll P$ such that X_t is a Q-martingale with respect to the enlarged filtration $\tilde{\mathcal{F}}_t$: more precisely

$$\frac{dQ}{dP} = G(\eta)\mathcal{E}\left(-\int_0^\cdot \lambda_s(\eta) dW_s\right)_T$$

with G such that the above expression is square integrable and $E[G]=1$. Q is a true probability if $G>0$.

Before we find an equation to identify $\widetilde{P}$, we need another

Definition 1.13. We define the two processes $\widehat{W}_t$ and W_t^* as follows:

$$\widehat{W}_t = W_t + \int_0^t \lambda_s ds$$

$$W_t^* = W_t + 2\int_0^t \lambda_s ds$$

Remark 1.14. By the theorem of Girsanov, if $\mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s\right)_t$ and $\mathcal{E}\left(-2\int_0^\cdot \lambda_s dW_s\right)_t$ are uniformly integrable martingales, then $\widehat{W}_t$ and W_t^* are Brownian Motions respectively under the measures $\widehat{P}$ and P^*, defined as

$$\frac{d\widehat{P}}{dP} = \mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_T \quad \text{and} \quad \frac{dP^*}{dP} = \mathcal{E}\left(-2\int_0^\cdot \lambda_t dW_t\right)_T$$

We recall that $\widehat{P}$ (if it exists) is called the *minimal martingale measure.*

Lemma 1.15. *Let h, k be two predictable stochastic processes whose stochastic integrals $\int_0^t h_s dW_s^*$ and $\int_0^t k_s dM_s$ are defined. The following conditions are equivalent:*

$$\exp\left(\int_0^T \lambda_s^2 ds\right) = c\,\frac{\mathcal{E}\left(\int_0^\cdot h_s dW_s^*\right)_T}{\mathcal{E}\left(\int_0^\cdot k_s dM_s\right)_T} \tag{2}$$

$$\mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s + \int_0^\cdot k_s dM_s\right)_T = c\,\mathcal{E}\left(\int_0^\cdot (-\lambda_s + h_s) d\widehat{W}_s\right)_T \tag{3}$$

where c is the same constant in both equations.

Proof. We will use the properties of the *Doléans exponential* listed in Protter (1990) pag. 79. Starting from the left-hand side of (3), we have

$$\mathcal{E}\left(-\int_0^\cdot \lambda_s dW_s + \int_0^\cdot k_s dM_s\right)_T =$$
$$= \mathcal{E}\left(-\int_0^\cdot \lambda_s d\widehat{W}_s\right)_T \mathcal{E}\left(\int_0^\cdot k_s dM_s\right)_T \exp\left(\int_0^T \lambda_s^2 ds\right)$$

Conversely, starting from the right-hand side of (3), we have

$$\mathcal{E}\left(\int_0^\cdot (-\lambda_s + h_s) d\widehat{W}_s\right)_T =$$
$$= \mathcal{E}\left(-\int_0^\cdot \lambda_s d\widehat{W}_s\right)_T \mathcal{E}\left(\int_0^\cdot h_s d\widehat{W}_s\right)_T \exp\left(\int_0^T \lambda_s h_s ds\right) =$$
$$= \mathcal{E}\left(-\int_0^\cdot \lambda_s d\widehat{W}_s\right)_T \mathcal{E}\left(\int_0^\cdot h_s dW_s^*\right)_T$$

The conclusion is now immediate. □

From now on, we suppose that $\mathcal{M}_e^2 \neq \emptyset$. By Schweizer (1996), Lemma 1 page 210, and Delbaen/Schachermayer (1996) lemma 2.2 and Theorem 1.3, we obtain the following characterization of the variance-optimal martingale measure: $\widetilde{P}$ is an element of $\mathcal{M}_e^2$ (i.e. $\widetilde{P}$ is a true probability) and it is the unique element of $\mathcal{M}_s^2$ which can be written in the form

$$\frac{d\widetilde{P}}{dP} = c + \int_0^T \gamma_s dX_s$$

with $c \geq 1$. In the above equation, γ_t is a predictable stochastic process which does not necessarily belong to Θ ; however the integral process $\int_0^t \gamma_s dX_s$ is a square integrable martingale for every probability measure $Q \in \mathcal{M}_e^2$. In particular, $\int_0^T \gamma_s dX_s$ is an element of $\overline{G_T(\Theta)}$.

Since $\dfrac{d\widetilde{P}}{dP}$ is strictly positive, it can be written as a *Doléans exponential.* From the previous result, we obtain the following :

Theorem 1.16. *Let h, k be two predictable processes such that the exponential martingale $\mathcal{E}\left(-\int_0^{\cdot} \lambda_s dW_s + \int_0^{\cdot} k_s dM_s\right)$ is square-integrable. Then h, k are solutions of the equation (2) of Lemma 1.15 if and only if*

$$\frac{d\widetilde{P}}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_s dW_s + \int_0^{\cdot} k_s dM_s\right)_T = \frac{\mathcal{E}\left(-\int_0^{\cdot} \beta_s dX_s\right)_T}{E\left[\mathcal{E}\left(-\int_0^{\cdot} \beta_s dX_s\right)_T\right]}$$

where $\beta_s = \dfrac{\lambda_s - h_s}{\sigma_s X_s}$.

The equality $\dfrac{d\widetilde{P}}{dP} = c\,\mathcal{E}\left(-\int_0^{\cdot} \beta_s dX_s\right)_T$ is useful to characterize the optimal strategy (see Rheinländer/Schweizer (1997)); we also recall that β is the so-called *hedging numéraire* of Gouriéroux et al. (1998).

2. Explicit Solutions

We have seen that a solution to the equation:

$$\exp\left(\int_0^T \lambda_s^2 ds\right) = c\frac{\mathcal{E}\left(\int_0^{\cdot} h_s dW_s^*\right)_T}{\mathcal{E}\left(\int_0^{\cdot} k_s dM_s\right)_T}$$

provides an explicit form for the density of the variance-optimal martingale measure.

Definition 2.1. We recall the definition of the *mean-variance tradeoff* process $\widehat{K}_t$ (see, for instance, Schweizer (1996)):

$$\widehat{K}_t = \int_0^t \lambda_s^2 ds$$

From (2) we can immediately see the following:

Proposition 2.2. *$\widehat{K}_T$ is a constant if and only if $\widetilde{P} = \widehat{P}$, and $\beta = \dfrac{\lambda}{\sigma X}$.*

This was first pointed out by Pham et al. (1998) and, for Itô processes, by Laurent/Pham (1999).

In more realistic situations, a solution to (2) can easily be found in two cases:

(α) $\lambda_s(\omega, \eta) = \lambda_s(\omega)$: in this case we set $k = 0$, and solve the equation

$$\mathcal{E}\left(\int_0^{\cdot} h_t dW_t^*\right)_T = \frac{\exp\left(\int_0^T \lambda_t^2 dt\right)}{E^*\left[\exp\left(\int_0^T \lambda_t^2 dt\right)\right]}$$

which, provided that E^* exists, and the above expectation is finite, admits a solution by the representation property of W (and thus of W^*) on Ω. This case covers the so-called *almost complete models*, where $\widetilde{P} = \widehat{P}$, while $\beta_s = \dfrac{\lambda_s - h_s}{\sigma_s}$.

In a typical example, H is an option on two observable assets, but trading is allowed in only one of them. As a result, $\mathcal{F}_t^S$ is strictly smaller than $\mathcal{F}_t^W \otimes \mathcal{E}_t$, unlike in the usual stochastic volatility models, where these filtrations are equal.

For a discussion on *almost complete models*, see for instance Pham et al. (1998) or Laurent/Pham (1999).

We only remark that in this case $\mathcal{M}_e^2 \neq \emptyset$ if and only if $\widehat{P}$ exists and $\frac{d\widehat{P}}{dP}$ is in L^2. Since $(\frac{d\widehat{P}}{dP})^2 = \mathcal{E}\left(-2\int_0^{\cdot} \lambda_s dX_s\right)_T \exp\left(\int_0^T \lambda_t^2 dt\right)$, this condition is satisfied if the probability P^* exists and $\exp\left(\int_0^T \lambda_t^2 dt\right)$ is P^*-integrable.

(β) $\lambda_s(\omega,\eta) = \lambda_s(\eta)$: in this case we can simply set $h = 0$, and then solve the equation

$$\mathcal{E}\left(\int_0^{\cdot} k_t dM_t\right)_T = \frac{\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)\right]}$$

which always admits a solution, since M has the representation property on E. This case covers all examples considered in Biagini/Guasoni (1999) : $\beta_s = \frac{\lambda_s}{\sigma_s X_s}$, and $\widetilde{P}$ is generally different than $\widehat{P}$, unless $\widehat{K}_T$ is deterministic (for diffusion processes, this is proved Pham et al. (1998) , Theorem 11).

We remark that if $\int_0^T \lambda_t^2(\eta)dt$ is finite almost surely, then $\mathcal{M}_e^2 \neq \emptyset$. Namely, in this case we obtain

$$(\frac{d\widetilde{P}}{dP})^2 = \mathcal{E}\left(-2\int_0^{\cdot} \lambda_t^2(\eta)dW_t\right)_T \frac{\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)\right]^2}$$

The process $\mathcal{E}\left(-2\int_0^{\cdot} \lambda_t^2 dW_t\right)_T$ is actually a stochastic integral depending on the parameter η (see Protter (1990) for details): therefore for every fixed η we have that $\int_\Omega \mathcal{E}\left(-2\int_0^{\cdot} \lambda_t^2(\eta)dW_t\right)_T dP(\omega) = 1$, and consequently we get

$$E\left[(\frac{d\widetilde{P}}{dP})^2\right] = \frac{1}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)\right]}$$

When $\lambda_s(\omega,\eta) = \lambda_s(\eta)$, it may be hard to find k explicitly ; but in fact it is often sufficient to know that it exists, since $\widetilde{P}$ can be obtained through the equality

$$\frac{d\widetilde{P}}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_s dW_s\right)_T \frac{\exp\left(-\int_0^T \lambda_t^2 dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2 dt\right)\right]} \tag{4}$$

Below we have some examples:

Example 2.3. If we consider the example 1.4 , under the probability $\widetilde{P}$ the numbers a_i are still independent , but $a_i = 1$ with a new probability $\tilde{p}$, where , if $\Delta T = \frac{T}{n+1}$,

$$\tilde{p} \ = \ \frac{p\, e^{-\lambda_1^2(\Delta T)}}{p\, e^{-\lambda_1^2(\Delta T)} + (1-p)\, e^{-\lambda_2^2(\Delta T)}}$$

Example 2.4. If we consider the example 1.6, under the new probability $\widetilde{P}$ the time jump τ and the new values of μ and σ after τ are no more independent: in Biagini/Guasoni (1999) one can find the explicit form of the law of τ under $\widetilde{P}$ and of the laws of μ and σ conditional to $\{\tau = t\}$.

In some models, however, it may be desirable to find k_t: this is the case, for instance, for stochastic volatility models defined by diffusion processes. In example 1.7, if $\beta(t,x,y) = 0$ and if α, γ, σ don't depend on X_t , we have $\lambda_s(\omega,\eta) = \lambda_s(\eta)$, and $\widetilde{P}$ can be written as in (4): however, this does not clarify the dynamics of v_t under $\widetilde{P}$. On the other hand, if k_t is known, then one can apply Girsanov's Theorem, and get

$$\begin{cases} dX_t = X_t\sigma(t,v_t)d\widetilde{W}_t^1 \\ dv_t = (\alpha(t,v_t) - k_t)dt + \gamma(t,v_t)d\widetilde{W}_t^2 \end{cases}$$

where $\widetilde{W}_t^1$ and $\widetilde{W}_t^2$ are independent Wiener processes under $\widetilde{P}$.

If the model is in some sense "Markovian", we obtain the following result (which coincides with Proposition 6.1 (3) of Laurent/Pham (1999) , but it is proved in a completely different way) :

Proposition 2.5. *Assume that* $E\left[\exp\left(-\int_t^T \lambda_s^2(s,v_s)ds\right)\middle|\,\mathcal{F}_t\right] = G(t,v_t)$, *and that the function* $G(t,x)$ *is* C^1 *in* t *and* C^2 *in* x*. Then we have*

$$\mathcal{E}\left(\int_0^\cdot k_s dW_s^2\right) = \frac{\exp\left(-\int_0^T \lambda^2(s,v_s)ds\right)}{E\left[\exp\left(-\int_0^T \lambda^2(s,v_s)ds\right)\right]} \qquad \textit{iff} \qquad k_t = \left.\frac{\frac{\partial G}{\partial x}\gamma}{G}\right|_{(t,v_t)}$$

Proof. By martingale representation, there exists a process g_t such that

$$\exp\left(-\int_0^T \lambda^2(s,v_s)ds\right) = G_0 + \int_0^T g_s dW_s^2$$

Therefore:

$$\begin{aligned} G_t = E\left[G_T \middle|\, \mathcal{F}_t\right] = G_0 + \int_0^t g_s dW_s^2 = \\ = \exp\left(-\int_0^t \lambda^2(s,v_s)ds\right) E\left[\exp\left(-\int_t^T \lambda^2(s,v_s)ds\right)\middle|\,\mathcal{F}_t\right] = \\ = \exp\left(-\int_0^t \lambda^2(s,v_s)ds\right) G(t,v_t) \end{aligned}$$

Applying Itô's formula, we obtain:

$$dG_t = g_t dW_t^2 = \exp\left(-\int_0^t \lambda^2(s,v_s)ds\right)\left(\frac{\partial G}{\partial x}\gamma\right)(t,v_t)dW_t^2$$

where, in the last equality, the sum of the terms of finite variation vanishes since G_t is a martingale. Therefore, $g_t = \exp\left(-\int_0^t \lambda^2(s,v_s)ds\right)\left(\frac{\partial G}{\partial x}\gamma\right)(t,v_t)$. However, we also have

$$G_T = G_0 + \int_0^T g_s dW_s^2 = G_0 \mathcal{E}\left(\int_0^\cdot \frac{g_s}{G_s} dW_s^2\right)_T = G_0 \mathcal{E}\left(\int_0^\cdot k_s dW_s^2\right)_T$$

therefore $k_t = \dfrac{g_t}{G_t}$, and the proof is complete. □

3. Conditions for the closedness of $G_T(\Theta)$

The closedness of the space $G_T(\Theta)$ in $L^2(P)$ plays a key role in mean-variance hedging, since it guarantees the existence of an optimal hedging strategy in the space Θ.

A sufficient condition for $G_T(\Theta)$ to be closed is the boundedness of $\widehat{K}_T$, as shown by Pham et al. (1998). In some sense, we now show that in cases (α) and (β), the boundedness of $\widehat{K}_T$ is almost necessary. We will show that this condition is not satisfied for some commonly used models.

First we recall, and state as a theorem, a short version of a necessary and sufficient condition established by Delbaen et al. (1997) :

Theorem 3.1. *Let X be a continuous semimartingale: suppose that $\mathcal{M}_e^2 \neq \emptyset$ and let $Z_t = E\left[\frac{d\tilde{P}}{dP}\middle| \mathcal{F}_t\right]$. The following conditions are equivalent:*

i) $G_T(\Theta)$ is closed in $L^2(P)$;

ii) Z_t satisfies the following reverse Hölder inequality:

$$E\left[\left(\frac{Z_T}{Z_\tau}\right)^2 \middle| \mathcal{F}_\tau\right] \leq C$$

for all stopping times $\tau \leq T$ and for some constant C.

We shall now see how this condition translates for (α) and (β).

Proposition 3.2. *Assume that $\mathcal{M}_e^2 \neq \emptyset$:*

i) If $\lambda_s(\omega, \eta) = \lambda_s(\omega)$, then $G_T(\Theta)$ is closed if and only if there exists some M such that, for all stopping times τ,

$$E^*\left[\exp\left(\int_\tau^T \lambda_t^2(\omega)dt\right)\middle| \mathcal{F}_\tau\right] < M$$

ii) If $\lambda_s(\omega, \eta) = \lambda_s(\eta)$, then $G_T(\Theta)$ is closed if and only if there exists some $\epsilon > 0$ such that, for all stopping times τ,

$$E\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle| \mathcal{F}_\tau\right] > \epsilon$$

Proof. From 3.1, it follows that $G_T(\Theta)$ is closed if and only if condition $ii)$ in Theorem 3.1 is satisfied.

For $ii)$, we have

$$Z_\tau = E\left[Z_T \middle| \mathcal{F}_\tau\right] = \mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_\tau \frac{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)\middle| \mathcal{F}_\tau\right]}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)\right]} =$$

It follows that

$$\frac{Z_T}{Z_\tau} = \frac{\mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_T}{\mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_\tau} \frac{\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]} =$$

$$= \mathcal{E}\left(-\int_\tau^\cdot \lambda_t dW_t\right)_T \frac{\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)}{E\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]}$$

Therefore:

$$E\left[\left(\frac{Z_T}{Z_\tau}\right)^2\middle|\mathcal{F}_\tau\right] = \frac{E\left[\mathcal{E}\left(-2\int_\tau^\cdot \lambda_t dW_t\right)_T \exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]}{E\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]^2} =$$

$$= \frac{E^*\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]}{E\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]^2}$$

However, since λ depends only on η, we find that the projection of P^* on $\mathcal{F}_E$ coincides with P, and thus $E^*\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right] = E\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]$. Hence

$$E\left[\left(\frac{Z_T}{Z_\tau}\right)^2\middle|\mathcal{F}_\tau\right] = \frac{1}{E\left[\exp\left(-\int_\tau^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_\tau\right]}$$

as claimed. For i), calculations are more straightforward:

$$Z_\tau = \mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_\tau$$

and thus

$$\frac{Z_T}{Z_\tau} = \frac{\mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_T}{\mathcal{E}\left(-\int_0^\cdot \lambda_t dW_t\right)_\tau} = \mathcal{E}\left(-\int_\tau^\cdot \lambda_t dW_t\right)_T$$

Finally

$$E\left[\left(\frac{Z_T}{Z_\tau}\right)^2\middle|\mathcal{F}_\tau\right] = E\left[\mathcal{E}\left(-2\int_\tau^\cdot \lambda_t dW_t\right)_T \exp\left(\int_\tau^T \lambda_t^2(\omega)dt\right)\middle|\mathcal{F}_\tau\right] =$$

$$= E^*\left[\exp\left(\int_\tau^T \lambda_t^2(\omega)dt\right)\middle|\mathcal{F}_\tau\right]$$

and the proof is complete. □

We shall give some models where $G_T(\Theta)$ is not closed.

Example 3.3. Consider example 1.3 (or better the generalization of Föllmer and Leukert) , where:

$$\lambda_t = \lambda\, 1_{\{t<t_0\}} + \lambda(\eta)1_{\{t\geq t_0\}}$$

As mentioned before, here $E = \mathbb{R}$: $G_T(\Theta)$ is closed if and only if the distribution of $\lambda(\eta)$ has compact support.

In fact, if last condition is satisfied, then $\widehat{K}_T$ is bounded ; conversely, for $t \geq t_0$ we have

$$E\left[\exp\left(-\int_t^T \lambda_s^2(\eta)ds\right)\middle|\mathcal{F}_t\right] = \exp\big(-(T-t)\lambda^2(\eta)\big)$$

By Proposition 3.2 , the conclusion is immediate.

Example 3.4. We now examine the Heston model, that is a stochastic volatility model described by the equations

$$\begin{cases} dX_t \;=\; X_t\big(\lambda_0 v_t dt + \sqrt{v_t}dW_t^1\big) \\ dv_t \;=\; \big(\alpha - \beta v_t\big)dt \;+\; \sqrt{v_t}dW_t^2 \end{cases}$$

Here we have (see, for instance, Laurent/Pham (1999)) :

$$E\left[\exp\left(-\int_t^T \lambda_t^2(\eta)dt\right)\middle|\mathcal{F}_t\right] = \exp\big(-A(T-t)\lambda_0^2 v_t - B(T-t)\big) \tag{5}$$

where

$$A(\tau) = \frac{1+\zeta}{\delta}\frac{1-e^{-\delta\tau}}{1+\zeta e^{-\delta\tau}} \qquad \delta = \beta\sqrt{1+\frac{2\lambda_0^2}{\beta^2}} \qquad \zeta = \frac{\delta-\beta}{\delta+\beta}$$

Since $\delta, \zeta > 0$, it follows that $A(T-t) > 0$, and therefore (5) is bounded from below if and only if v_t is bounded from above. However, this is never the case, since in Heston model v_t is the square of a Bessel process with an appropriate change of time.

Analogous calculations can be carried out in the Stein and Stein model (see Heath et al. (1999), example 3.2.2 for details) showing that also in this case $G_T(\Theta)$ is not closed.

We point out that the drawback of the non-closedness of the space $G_T(\Theta)$ has been overcome by Schweizer (1999) : by exploiting the approach introduced by Gouriéroux et al. (1998) , Schweizer has proved the existence of an optimal mean-variance strategy not in the space Θ , but in the space $\widetilde{\Theta}$ of all predictable processes θ such that the stochastic integral $\int_0^t \theta_s dX_s$ is a square integrable martingale for every $Q \in \mathcal{M}_e^2$.

4. Conclusions

We have seen that a simple equation involving stochastic exponentials can identify the variance optimal probability $\widetilde{P}$ (and the mean-variance hedging strategy) in a general class of stochastic volatility models. All examples introduced are analysed in Biagini/Guasoni (1999).

We further point out that the *change of numéraire* approach introduced by Geman et al. (1995) can be adapted to give the explicit form of the mean-variance hedging strategy for a call option (see Biagini/Guasoni (1999)).

5. Appendix

Proposition 5.1. *Any square-integrable martingale with respect to the filtration $\tilde{\mathcal{F}}_t$ can be written as*

$$N_t(\omega,\eta) = N_0(\eta) + \int_0^t H_s(\omega,\eta)dW_s(\omega)$$

where H is $\tilde{\mathcal{F}}_t$-predictable and such that $E\left[\int_0^T H_s^2 ds\right] < \infty$.

Proof. Denote by $\mathfrak{M}$ the set of martingales which admit a representation in the desired form. We begin by showing that $\mathfrak{M}$ contains all martingales N_t such that $N_T(\omega,\eta) = F(\omega)G(\eta)$, with F, G square-integrable and measurable functions. In fact, if $F(\omega) = F_0 + \int_0^T H_s(\omega)dW_s(\omega)$, with $E\left[F^2\right] = F_0^2 + E\left[\int_0^T H_s^2 ds\right]$, it is easily seen that:

$$F(\omega)G(\eta) = F_0 G(\eta) + \int_0^T H_s(\omega)G(\eta)dW_s(\omega)$$

The stochastic process $\widetilde{H}_s(\omega,\eta) = H_s(\omega)G(\eta)$ is $\tilde{\mathcal{F}}_t$-predictable, and

$$E\left[F^2G^2\right] = E\left[F_0^2G^2\right] + E\left[\int_0^T H_s^2 G^2 ds\right]$$

$\mathfrak{M}$ is obvoiusly stable under linear combinations, hence the set $\{N_T : \ N \in \mathfrak{M}\}$ is dense in $L^2(\Omega \times E, \mathcal{F}_T \otimes \mathcal{E}, P)$. However, if $N_t = N_0 + \int_0^t H_s dW_s$, the map $N \mapsto E\left[N_T^2\right] = E\left[N_0^2 + \int_0^T H_s^2 ds\right]$ is an isometric injection from $\mathfrak{M}$ into $L^2(\Omega \times E, \mathcal{F}_T \otimes \mathcal{E}, P)$. This concludes the proof. □

Proposition 5.2. *Any square-integrable martingale N_t with respect to the filtration $\mathcal{F}_t^W \otimes \mathcal{E}_t = \mathcal{F}_t$ can be written in the form*

$$N_t(\omega,\eta) = N_0 + \int_0^t H_s(\omega,\eta)dW_s(\omega) + \int_0^t K_s(\omega,\eta)dM_s(\eta)$$

with H, K predictable and such that:

$$E\left[\int_0^T H_s^2 ds + \int_0^T K_s^2 d[M]_s\right] < \infty$$

Proof. Denote by $\mathfrak{M}$ the set of martingales which admit a representation in the desired form. We start showing that $\mathfrak{M}$ contains all martingales N_t such that $N_T(\omega,\eta) = F(\omega)G(\eta)$. We write $F(\omega) = F_0 + \int_0^T H_s(\omega)dW_s(\omega)$, $G(\eta) = G_0 + \int_0^T K_s(\eta)dM_s(\eta)$, and consider the martingales $R_t = E\left[F|\, \mathcal{F}_t\right]$ and $V_t = E\left[G|\, \mathcal{F}_t\right]$. By Itô's formula, and recalling that $[W, M] = 0$, we have

$$F(\omega)G(\eta) = F_0 G_0 + \int_0^T V_s H_s dW_s + \int_0^T R_s K_s dM_s$$

Again, $\mathfrak{M}$ is stable under linear combinations, and $\{N_T : \ N \in \mathfrak{M}\}$ is dense in L^2. Since the map $N \mapsto E\left[N_0^2 + \int_0^T H_s^2 ds + \int_0^T K_s^2 d[M]_s\right]$ is an isometric injection from $\mathfrak{M}$ in L^2, the proof is complete. □

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